ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645
Indian Journal of Science and Technology, *Vol 9(45), DOI: 10.17485/ijst/2016/v9i45/105079, December 2016*

Radial Basis Function Methods for Solving Partial Differential Equations-A Review

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Abstract

Background/Objectives: The approximation using radial basis function (RBF) is an extremely powerful method to solve partial differential equations (PDEs). This paper presents different types of RBF methods to solve PDEs. **Methods/ Statistical Analysis:** Due to their meshfree nature, ease of implementation and independence of dimension, RBF methods are popular to solve PDEs. In this paper we examine different generalized RBF methods, including Kansa method, Hermite symmetric approach, localized and hybrid methods. We also discussed the preference of using meshfree methods like RBF over the mesh based methods. **Findings:** This paper presents a state-of-the-art review of the RBF methods. Some recent development of RBF approximation in solving PDEs is also discussed. The mathematical formulation of different RBF methods are discussed for better understanding. RBF methods have been actively developed over the years from global to local approximation and then to hybrid methods. Hybrid RBF methods help in reduction of computational cost and become very effective in solving large scale problems. **Application/Improvements:** RBF methods have been applied to various diverse fields like image processing, geo-modeling, pricing option and neural network etc.

Keywords: Differential Quadrature Radial Basis Function, Kansa Collocation Method, Partition of unity, Partial Differential Equation, Radial Basis Function

1. Introduction

Most of the problems in biological, chemical, mechanical, electrical science can often be represented by partial differential equations. Now a days, researchers are using numerical techniques for solving different types of PDEs when analytically obtaining their exact solution is difficult. In all traditional numerical techniques for approximating the solution, data must be in predetermined pattern and contained in a simple region. In some problems, this condition cannot be met and traditional numerical techniques are not applicable on such problems. To overcome this problem, meshfree methods are used. One such method is Radial Basis function (RBF) collocation method.

The RBF methodology was first introduced by Hardy in 1971 in connection with a topological application on quadric surfaces[1]. He introduced the multiquadric (MQ) approximation scheme. Richard Franke in 1982, test rectangle based blending methods, inverse distance weighted methods, finite element based methods, foley's methods, global basis function type methods and modified maude methods on scattered data interpolation. He evaluates methods based on different parameters like accuracy, storage and time taken by the method and finds multiquadric (one of the RBF) among the best ones[2]. Micchelli in 1986 progressed further by proving that the multiquadraic surface interpolation is always solvable[3]. Edward Kansa in 1990, firstly use the multiquadraic, a globally supported interpolant to solve a PDE known as Kansa method[4,5].

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However applied to various applications, Kansa method has some disadvantage like unsymmetrical nature of interpolation matrix which leads to ill conditioned matrix for large number of nodes. In 1996, Fasshauer proposed a hermite based approach as modification of Kansa method. The collocation matrices from this approach are symmetric in nature and generally have smaller condition number[6]. However, the symmetric RBF collocation approach has its own limitations. Symmetric Collocation approach is difficult to implement as compared to unsymmetric approach. The comparison between symmetric and unsymmetric approach was done by Power and Barraco[7] and Larsson and Fornberg[8]. Several other procedures have been proposed to address the above difficulties like preconditioning the interpolation matrix[9], domain decomposition method[10,11] etc. Using these techniques, the ill conditioning of the matrix can be reduced to some extent. Other very promising approach to deal with these kinds of difficulties in the Kansa method is the local approach. In this approach, instead of using all the nodes in the whole domain, only the local approximation is to be considered for collocation. In 2004, diffusion problem has been solved by Chantasiriwan using local RBF[12]. In the continuation localised RBF methods are used by researchers to solve different PDEs[13-21].

Mostly RBFs are associated with a parameter which decide the shape of the RBF known as shape parameter discussed in section (2.3). Some of the RBFs give best accuracy with small value of shape parameter which leads to ill conditioning of the matrix. Fornberg and Wright in 2004 proposed an algorithm for stable computation of the RBF for all values of the shape parameter[22]. Fornberg and Piret in 2007 further improved the approach to a new approach RBF-QR which entirely eliminates ill-conditioning of the matrix in case of near flat basis function[23]. Further the method is developed by combining RBF with other known methods to get the best out of the RBFs. Shu in 2003 gave an approach to combine the meshfree nature of RBF and the high accuracy and simplicity of Differential Quadrature (DQ) method by proposing a hybrid method known as RBF-DQ method[24]. This technique has been used by researchers to solve PDEs in fluids (such as Navier-stokes, Shallow water problems). Tolstykh in 2003 used local set of nodes to generate the radial basis finite difference approach[25], this hybrid method termed as RBF-FD. For more literature on RBF-FD reader can refer[26-30]. Another promising approach is the RBF-PUM to solve PDEs which combines the partition of unity method with RBF[31]. The idea of RBF-PUM method is to partition the domain into overlapping subdomains. The local approximation is done on the subdomains and combines to get the global approximation. RBF-PUM reduces the computational cost while maintaining high accuracy.

This paper presents a review of the RBF methods. In section 2, the basic definition and concepts are presented. In section 3, a review of the development of the methods is presented. Last section represents the concluding part of the paper. We tried to give the mathematical formulation of the methods wherever possible. In authors' knowledge, there is no such survey available in which all the methods related to RBF are presented. A chronological summary of various methods along with their associated researcher is presented by Table 1 and Figure 1 shows a summary of development of RBF methods.

2. Radial Basis Function

2.1 Definition

A function $\Phi: \mathbf{R}^{\tau} \rightarrow \mathbf{R}$ is called radial if there exists a one variable function $\varphi : [0, \infty) \rightarrow \mathbf{R}$ such that

$\Phi(x) = \varphi(\|x\|)$, where $\|\cdot\|$ is the Euclidean norm.

2.2 Definition

A radial basis function $\varphi(r)$ is a univariate continuous real valued function which depends on the distance from the origin (or any other fixed centre point).

2.3 Some Important Commonly Used RBF

RBFs are mostly identified on the basis of smoothness. Some functions are infinitely smooth and some are piecewise smooth. Gaussian Function (GS), Multiquadric (MQ), Inverse Multiquadric (IMQ) and Inverse quadric (IQ) are some example of infinitely smooth RBFs where as Thin Plate Spline (TPS) and Linear radial function (LR) are piecewise smooth RBFs. For infinitely smooth

Table 1. A chronological scheme of RBF methods

Year	Methods	Associated Researcher	Reference
1990	Radial Basis Function Kansa Method	Edward Kansa	[4,5]
1997	Hermite Symmetric RBF-Collocation Method	Greg Fasshauer	[6]
2001	Modified Kansa Method (MKM)	W.Chen	[32]
2002	Radial Basis Function Differential Quadrature Method (RBF-DQ)	Shu Chang	[42]
2002	Radial Basis Function Partition Of Unity Method(RBF-PUM)	H.Wendland	[31]
2006	Local Radial Basis Function Collocation Method(LRBFCM)	B.Šarler and R.Vertnik	[13]
2007	Radial Basis Function-QR Method(RBF-QR)	B.Fornberg and	[23]
2011	Radial Basis Function-Finite Difference Method(RBF-FD)	G.Wright A.R. Tolstykh	[9]

Table 2. Some commonly used RBFs

Name of the RBF	Equation ($r = \|x\|$)
Gaussian Function (GS)	$\varphi(r) = e^{-(\epsilon r)^2}$
Linear radial function (LR)	$\varphi(r) = r$
Multiquadric (MQ)	$\varphi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse quadric (IQ)	$\varphi(r) = \frac{1}{1 + (\epsilon r)^2}$
Polyharmonic Spline(PHS)	$\varphi(r) = \begin{cases} r^{2k-1}; & k \in N \\ r^{2k} \ln(r); & k \in N \end{cases}$
Thin Plate Spline (TPS)	$\varphi(r) = r^2 \ln(r) \varphi(r) = r^2 \ln(r)$
Inverse Multiquadric (IMQ)	$\varphi(r) = \frac{1}{\sqrt{1 + (\epsilon r)^2}} \varphi(r) = \frac{1}{\sqrt{1 + (\epsilon r)^2}}$

RBFs, there exists a free parameter $\epsilon > 0$ called the shape parameter which controls the shape of RBF. The RBF become flat if shape parameter is closer to 0. Table 2, represents some RBFs which are commonly used.

2.4 Formation of Interpolation Problem

The interpolation problem is to find a smooth function

$$S(x_i) = f_i \quad for\ i = 1,2,3 \ldots, N \qquad (1)$$

for the given data

$$(x_i, f_i) \quad with\ x_i = 1,2,3 \ldots, N, x_i \in \mathbf{R}^s \text{ and } f_i \in \mathbf{R}$$

.

A RBF interpolant takes the form

$$S(x) = \sum_{i=1}^{n} \alpha_i \varphi(\|x - x_i\|)$$

We can obtain α_i by imposing the interpolation condition equation (1) which leads to

$$A\alpha = f \ .$$

where

$$A_{ij} = \|x_i - x_j\| \quad i,j = 1,2,3 \ldots, N$$

$$f = [f(x_1), f(x_2), f(x_3) \ldots, f(x_N)]^T$$

$$\alpha = [\alpha(x_1), \alpha(x_2), \alpha(x_3), \ldots, \alpha(x_N)]^T$$

The matrix A is called the interpolation matrix

3. Important RBF Methods For Solving PDEs

3.1 Kansa Method

Kansa method or RBF collocation method is one of the meshless methods. Meshless methods have great

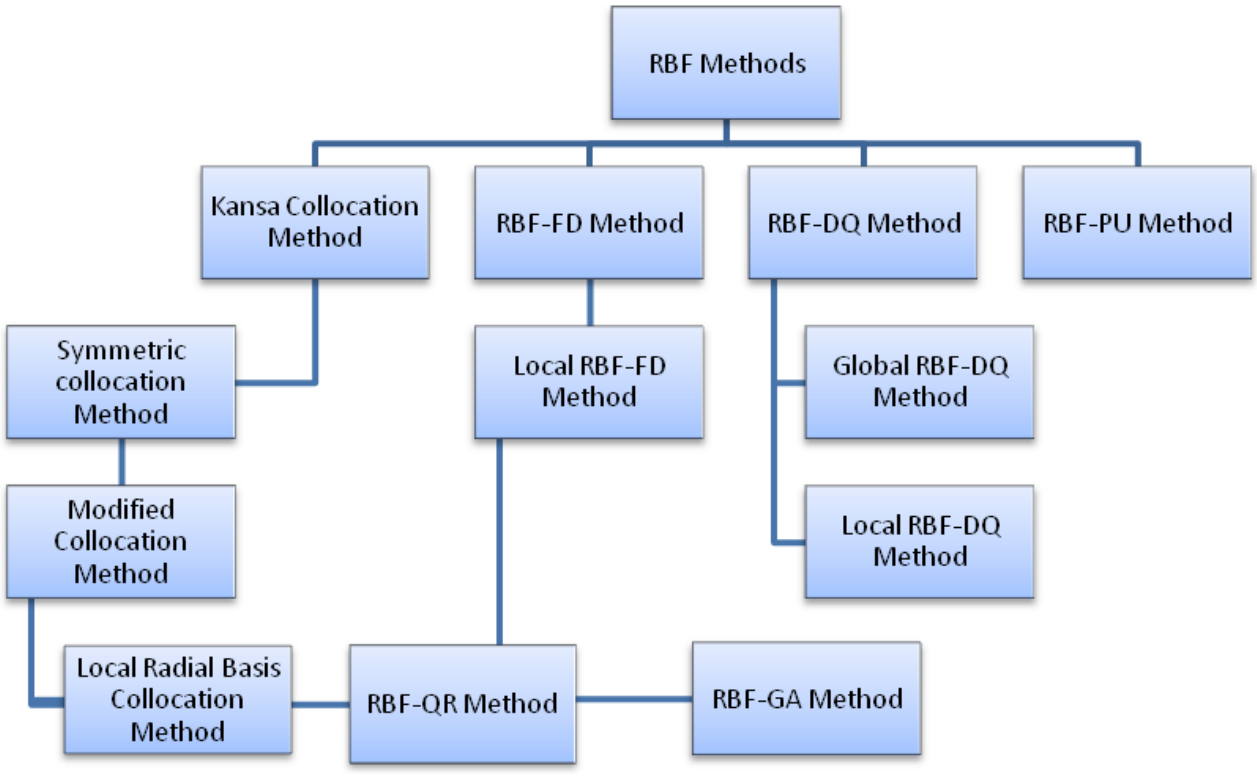

Figure 1. A Summary of Development of RBF-Methods.

advantage over the mesh methods. It requires neither domain nor surface discretization hence they are cost effective. Kansa[4-5] in 1990, proposed an asymmetric method to solve PDEs using RBFs known as Kansa method. For mathematical formulation, let us consider the elliptic partial differential equation with $\Omega \subset \mathbf{R}^{\tau}$

$$L[u(x)] = f(x), x \in \Omega \quad (2)$$

with boundary condition $u(x) = g(x), x \in \partial\Omega$ (3)

Let $\{x_j\}$, j = 1,2,...,N_i be the points in the whole domain Ω and $\{x_j\}$, j = $N_{i+1}, \dots, N$ be the points on the boundary $\partial\Omega$. This method assume to represent the solution $u(x)u(x)$ by a linear combination of RBFs at predefined nodes

$$u(x) = \sum_{j=1}^{N} \alpha_j \varphi(\|x - x_j\|_2) \quad (4)$$

where α_j are unknown coefficients to be determined and *N* is the total number of nodes. φ is the radial basis function such as MQ, IMQ, IQ etc as mentioned in Table 2. Substituting equation (4) in equation (2) and equation (3) gives the linear system of equations

$$A\alpha = B \quad (5)$$

Where

$$\alpha = [\alpha(x_1), \alpha(x_2), \alpha(x_3), \dots \dots \dots \dots \alpha(x_N)]^T,$$

and

$$A = \begin{bmatrix} L(\varphi(\|x - x_j\|_2))|_{(x = x_i)} \\ \varphi(\|x - x_j\|_2)|_{(x = x_i)} \end{bmatrix}$$

This approach has been widely successful in solving various problems. Chen at el. solved convection diffusion problems[32], Using this approach, Stefan problem is solved by Šarler[33], Zhou solved shallow water modelling problem[34], Chantasiriwan solved time dependent heat conduction problems[35], Duan at el. used Kansa method to solve electrostatic problems[36], Chen solved fractional diffusion equation by Kansa method[37]. Although used to solve various PDEs, the Kansa methods have certain disadvantages also. One of the biggest problem in this method is that it produces unsymmetric interpolation matrix and hence the computational cost of the method becomes very high. In a region adjacent to the boundary, the accuracy of the method is lower. The easiest way to increase the accuracy is to increase the number of interpolation points which results in high condition number of the matrix. Thus the method is not useful for large scale problems where the interpolation points are large. When we take the collocation points in the whole physical domain, the resultant matrix become dense and thus ill conditioned. Further, it is still a difficult task to find the best shape parameter of various RBFs.

The Kansa method is further upgraded to a method known as **Symmetric collocation method**. Fasshauer[6] in 1997 proposed this method which is based on Hermite interpolation. The RBF expansion for approximating the function u as proposed by Fasshauer is given by:

$$\bar{u}(x) = \sum_{j=1}^{N_I} \alpha_j L^* \varphi\left(\|x - \xi_j\|\right) + \sum_{j=N_{I+1}}^{N} \alpha_j \varphi\left(\|x - \xi_j\|\right)$$

where the number of interior nodes of Ω is N_I , $\{\xi_1, \xi_2, \xi_3 \dots \dots, \xi_N\}$ are the points known as centres of RBFand L^* is the differential operator in equation (2). After applying the governing equations i.e collocation conditions, we will have an *N* X *N* symmetric collocation matrix. For appropriate value of φ the matrix is non-singular. The above discussed symmetric and unsymmetric approach has been implemented for many different applications. Power and Barraco[7] compared these methods and find that the symmetric approach is slightly better than the unsymmertic (Kansa) approach. The computational cost is less in case of symmetric approach but the Kansa method is easy to implement. Leitao[38] used the symmetric approach to solve 2D elastostatic problems. Rocca et al.[39-42] solved some time dependent PDEs and Naffa[43] used it to solve non-linear plate problems. Chen[44] proposed a

method known as Modified Collocation Method (MCM) which is based upon Green second identity. As the MCM is the modified form of symmetric collocation method, so the interpolation matrix is also symmetric.

While using the Kansa collocation method for solving the PDEs, ill conditioning of the interpolation matrix is the serious issue. Several remedies were proposed to solve this like preconditioning, Domain Decomposition Method (DDM) and compactly supported RBF. Preconditioning means transforming a system of linear equation $Ax = B$ into another system, which is more favourable for iterative solution. A matrix that produces such a transformation is known as preconditioner. Preconditioning helps in improving the convergence of the methods and also takes care high value of condition number. Whereas, the Domain Decomposition method is to subdivide a problem with large number of points into subdomains either overlapping or non-overlapping. The method is very useful in avoiding the ill-conditioning by solving small subdomain problems instead of one large domain problem.

Kansa and Ling[9] in 2005 developed an effective preconditioning scheme based on least square construction of the approximate cardinal basis function (ACBF). Ideally ACBF is equivalent to a delta function i.e one at its centre and zero elsewhere. They proposed the scheme for asymmetric collocation RBF methods. The least square construction of the preconditioner, makes the method simple and inexpensive. Brown et al.[45] further review the ACBF preconditioning technique for interpolation problem and compared the technique by applying it to study state problems. He concluded that this technique is a good choice for ill-conditioned problems. Kansa and Ling[10] combined the ACBF preconditioning technique with DDM. They use the classic alternating Schwarz algorithm approach for DDM. The rank of each subdomain matrix is reduced using DDM and the preconditioning becomes more effective. This method reduces the overall computational cost and become more effective in solving RBF. Li and Hon[11] tried the DDM to solve stationary PDEs. On matching and non matching grid points, they developed both overlapping and non overlapping DDMs. With the help of numerical examples, they show that the non matching grid and matching grid achieve the same accuracy with the same iteration steps.

Apart from these methods there is another approach to avoid the dense nature of the interpolation matrix. The approach is based on local approximation and known as Local Radial Basis Function Collocation Method (LRBFCM). The method is described by Chen[46]. For mathematical formulation, consider the PDE given by equation (2) and (3). Let $u(x^s)$ is the local approximation of the solution $u(x)$ and $\{x^s\}_{s=1}^{N} \in \Omega$ then

$$\tilde{u}(x^s) = \sum_{k=1}^{n} [\alpha_k^s \varphi(\|x^s - x_k^s\|) \tag{6}$$

where x^s is the collocation point, n is the number of neighbouring points $\{x_k^s\}_{k=1}^{n}$ surrounding the point x^s including itself, φ is an RBF and $\{\alpha_1 k^1 s\}$ coefficients to be determined. If all the collocations points are distinct then the matrix

$$\Phi = [\varphi(\|x_1 i^1 s - x_1 j^1 s\|)]_{1}(i,j = 1)^1 n$$

will be non-singular if $\varphi(x)$ be strictly positive definite function[46]. Here the unknown coefficients are determined with the help of the equation (6)

$$\tilde{u}(x_i^s) = \sum_{k=1}^{n} [\alpha_k^s \varphi\left(\|x_i^s - x_k^s\|\right) \qquad i = 1,2,\dots,n$$

The above equation can be rewritten as

$$\alpha^s = \Phi^{-1}\mathbf{u}^s$$

where

$$\mathbf{u}^s = [\mathbf{u}^s(x_1^s), \mathbf{u}^s(x_2^s), \dots, \mathbf{u}^s(x_n^s)]^T \text{ and}$$

$$\alpha^s = [\alpha_1^s, \alpha_2^s, \dots, \alpha_n^s]$$

Now the approximate solution $\tilde{u}(x^s)$ can be rewritten in terms of given nodal values $[u(x]_i^s)$ at its n nearest neighbouring points

$$\tilde{u}^s = \Phi^s \alpha^s = \Phi^s [\Phi^{-1} u^s] = \Psi^s u^s \quad (7)$$

where $\Psi^s = \Phi^s \Phi^{-1} = \{\psi_j^s\}$. Rewriting the equation (7) in terms of approximate solution gives

$$\tilde{u}^s = \Psi u \quad (8)$$

where Ψ is sparse matrix of order N having N x n non zero elements. Substituting the value from equation (8) in equation (2) gives the linear system of equations which are sparse. We will get the approximate solution $\tilde{u}$ at all collocation points by solving the linear system of equations.

Šarler and Vertnik[13] inspired from the concept of local collocation used in[19,25] proposed the above discussed LRBFCM and applied this to find the solution of diffusion equations. The collocation is made locally on the overlapping domains of influence and hence reduces the collocation matrix size. Many authors followed this approach and applied this method to solve large scale complex problems such as convective-diffusive solid liquid phase change problems[14], fluid flow and heat transfer problems[15], Transport Phenomena[16] and Darcy flow[17] etc. This method is further improved by Siraj-Ul-Islam et al.[47] for the numerical solution of hyperbolic partial differential equation. In the improved approach, MQ is used as an RBF along with uniform nodal arrangement for better accuracy. The first order finite difference formula is used for approximating time derivative. Numerical problems shows that the method is more stable as compared to Kansa collocation method. The stability is established with the help of adaptive upwind technique. Mavric and Šarler[48] developed the LRBFCM for thermo elasticity problems and its error behaviour. The authors discussed the linear thermo-elastic problems. They considered the condition number of the collocation matrix as free parameter and shape parameter is used to achieve the desired condition number. This helps in improving the performance of even the non uniform node arrangement. Hon et al.[49] applied LRBFCM for solving coupled heat transfer and fluid flow problems with a free surface. The authors numerically solved the coupled mass, momentum and energy equations. Dehghan et al.[50] applied LRBFCM to solve keller-segel model for chemotaxis. Chemotaxis is the movement of cells or organisms in response to chemical gradients. The authors considered the keller-segel model that is represented by two non-linear PDEs, one for the cell density and other for chemoattractant concentration. First Crank-Nicolson scheme is used for time discretization then local RBF collocation method is used to approximate the spatial derivative. Finally LU decomposition method is used to solve equations after collocation. In this method, the interpolation matrix becomes sparse and cost effective.

One of the advantages of using RBF approximation for solving PDEs is the exponential convergence. As the shape parameter of the basis function become flat, the resultant interpolation matrix become ill conditioned and exponential convergence may not be possible. As discussed in section 2.3, various types of RBFs associated with a shape parameter ϵ and the limit $\epsilon \to 0$ leads to ill conditioned interpolation matrix. Fornberg and Wright[22] proposed an algorithm which gives firm and accurate values of RBF for small values of shape parameter. This was the first step towards the stable computations for small ϵ and known as Contour-Pade algorithm. The key part of the algorithm is considering the values of the shape parameter as complex numbers i.e $\epsilon = a + ib$ and in a large area around $\epsilon = 0$, the RBF interpolant is a measomorphic function with singularities as poles only. The algorithm is based on Cauchy Integral formula and Pade approximation. The limitation of the above algorithm is that it can only work with the small number of nodes and its been tricky to find the poles.

Fornberg and Piret[23] introduced the concept of **RBF-QR** method and applied the method when the nodes scattered over the surface of a sphere. This was the second algorithm that gave stable computation when $\epsilon \to 0$. Some RBFs gives best result when shape parameters is small. As discussed earlier, when RBF shape parameter become small, the interpolation matrix become ill conditioned. The RBF-QR method eliminates the ill conditioning and can be applicable to large number of nodes. The main logic in the RBF-QR method is to use a good basis that can span the same space instead of using the bad basis (ill conditioned). Fornberg[51] extends the concept of RBF-QR formulation for the nodes in one

dimensional, two dimensional and even to three dimensional with Gaussian RBF. This algorithm is also stable for the small shape parameter up to $\epsilon \to 0$. The authors use the chebyshev polynomial as basis for improving the conditioning. The RBF-QR method gives new direction to all local based RBF approximation as with the help of the discussed approach; one can find the optimal shape parameter for even very small value.

As per the application of RBF-QR is concerned, Piret and Hanert[52] used the method to solve fractional diffusion equations. They discretized the space fractional diffusion equation of one dimension with RBF-QR method. The author uses RBF-QR technique to remove the ill conditioning. Further in the same direction, Dehghan and Ilati[53] solved one and two dimensional time dependent coupled Sine-Gordon equations. The authors use Crank-Nicolson technique for time discretization the coupled Sine-Gordon equations. The stability issues related to small value of the shape parameter is discussed with the help of RBF-QR method.

Dehghan et al.[54] developed a way to use Kansa method to solve anomalous fractional sub diffusion equations. Fractional differential equations have applications in various fields of science and engineering. The authors used the Riemann-Liouville type of fractional derivative of equations. They first discretize the time fractional derivatives and then approximate the spatial derivative by Kansa collocation method. Energy method (which is based on choosing the solution itself as the test function) is used to prove the stability and convergence of time discreitized scheme. Further Dehghan et al.[55] showed that the approach is even suitable for the non-linear time fractional PDEs. The authors solved Sine-Gordon and Klein-Gordon equation which appears in relativistic field theory. They obtained a time discrete scheme by transforming the equation into low order system of equations. For checking the convergence and stability of the scheme, energy method is used. Numerically one and two dimensional Sine Gordon and Kelin-Gordon equations are solved on irregular domains. As collocation is done by Kansa method, LU decomposition is used to avoid the ill conditioning of the coefficient matrix.

Chen et al.[56] numerically solved boundary value problem elliptic in nature using Kansa RBF method. As discussed, global RBF methods leads to ill conditioning of the resultant matrix which leads to stability problem and high computational cost. They used Kansa method to discretize the problem which leads to a system where matrix possesses block circulant structure. In their study, they used Fast Fourier transform as a matrix decomposition algorithm to solve the block circulant structure. They used the leave-one-out cross validation (LOOCV) technique for finding a good shape parameter.

3.2 RBF-DQ Method

3.2.1 *Diffenertial Quadrature (DQ) Method*

Bellman et al.[57] in 1970 proposed the concept of DQ in which the derivative of the function is approximated instead of the function. Following the idea of integral quadrature, in this method the partial derivative of a smooth function is approximated as a linear combination of weights and its functional values in the whole domain. The derivative value of $f^n(x)$ with respect to x at a point x_i is the linear combination of all values of its function in the global domain by

$$f^n(x_i) = \sum_{j=1}^{N} a_{ij}^n \, f(x_j) \qquad i = 1,2,3 \ldots\ldots N$$

where a_{ij}^n are weight coefficients. There are many approaches to find the weight coefficients such as Lagrange interpolation polynomials[58], Fourier series expansion[58], Moving least square[59].

3.2.2 *Radial Basis Function-Differential Quadrature (RBF-DQ) Approach*

Shu[60] in 2002 uses radial basis function in place of Lagrange's interpolation in conventional DQ method and termed it as RBF-DQ method. In RBF-DQ method, radial basis functions are used as base function and also for the computation of weighting coefficients. This method is also suitable for non-linear problems Shu[24,61] gives two versions of the RBF-DQ method: Global and Local. Global approach uses all the nodes in the domain to expand the derivative at a specified node. When large nodes are used, the ill conditioning problem arises and

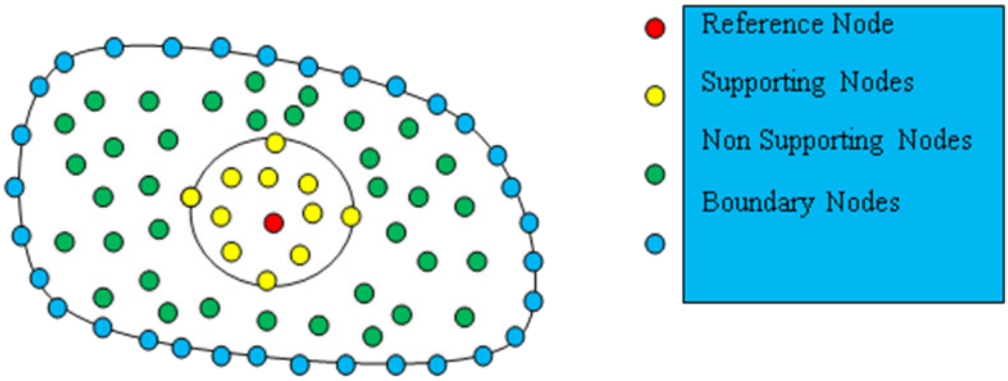

Figure 2. Supporting Nodes.

the computational cost becomes high. However, in local approach which is known as Local Radial Basis Function Differential Quadrature (LRBFDQ) method, only those nodes are used, which are in the neighbourhood of the node under consideration. These nodes are called supporting nodes as shown in Figure 2.

In this method, the derivatives are written as a linear sum of the values of the function at points in the domain under consideration. The n^{th} order derivative of a function $f(x,y)$ with respect to x at a certain node (x_i,y_i) is

$$f^n(x_i,y_i) \cong \sum_{j=1}^{N} w_{ij}^n \, f(x_j,y_j) \tag{9}$$

where N is the number of nodes in the supporting domain and w_{ij}^n are the weight coefficients at node (x_j,y_j) .To determine the weight coefficients, equation (9) must satisfied by the radial basis function $\varphi_k(x,y)$ as

$$\varphi_k{}^n(x_i,y_i) \cong \sum_{j=1}^{N} w_{ij}^n \, \varphi_k(x_j,y_j) \qquad \forall\, k = 1,2,3,\ldots,N \tag{10}$$

There are several kinds of RBFs as shown in Table 1. One can use any RBF which is infinitely smooth such as Multiquadric (MQ) $\varphi(r) = \sqrt{1+(\epsilon r)^2}$, Inverse Multi quadraic (IQ) $\varphi(r) = \dfrac{1}{1+(\epsilon r)^2}$, Gaussian (GA) $\varphi(r) = e^{-(\epsilon r)^2}$ etc. According to equation (10) at each point (x_i,y_i), we have linear system of equations as

$$\Phi_{x_i} = A W_{x_i}^n \tag{11}$$

where

Φ_{x_i} = EMBED Equation.DSMT4

A = EMBED Equation.DSMT4

and

$W_{x_i}^n$ = EMBED Equation.DSMT4

Therefore $W_{x_i}^n$ can be written as

$$W_{x_i}^n = A^{-1}\Phi_{x_i} \tag{12}$$

According to the equation (9), the n^{th} order derivative of function f with respect to x at node (x_i, y_i) can be expressed as

$$f^n(x_i, y_i) \cong W_{x_i}^{n\,T} F = \left[A^{-1}\Phi_{x_i}\right]^T F = \Phi_{x_i}{}^T \left[[A]^{-1}\right]^T F$$

where $F = [f_1, f_2, \ldots . f_N]^T$ and f_i is the value of f at node $(x_i, y_i)(x_i, y_i)$

Shu[24] solved two-dimensional incompressible Navier-Stokes equations using LRBFDQ method and further applied the method to compressible flows[62]. Shen[63] applied the method for the boundary layer problems. Soleimani et al.[64] solved two dimensional transient heat conduction problems. Shu[65] used indirect radial basis function network with differential quadrature and proposed method related to RBF-DQ named as **iRBF-DQ** method. He successively solved one dimensional burger's equation with the proposed method. Hashemi and Hatam[66] did seepage analysis using the method. The study of seepage is an important step in designing hydraulic structure. They proved that being its mesh independence property it works fine with irregular domains.

Dehghan[67] solved the boundary value problems using LRBFDQ of second order. He used Multiquadric (MQ) as basis function. As the shape parameter plays a very important role in RBF, Dehghan applied two different techniques to determine the optimal shape parameter OCSP technique and OVSP method.

Homayoon et al.[68] applied RBFDQ for simulation of long waves and shallow water equations. In this leave-one-out cross validation (LOOCV) technique is used to obtain the optimal shape parameter. Fantuzzi et al.[69] in 2015 proposed the application of RBFDQ method with domain decomposition technique and studied arbitrary shaped plates when composite materials are considered. Dehghan et al.[70] recently used this method to solve Cahn-Hilliard equations. He used Mutiquadric (MQ) as radial function with constant shape parameter and compared the method with global radial basis function (GRBF) method. He showed that the use of RBF-DQ reduces the ill conditioning problem upto certain extent in GRBF method.

Parand and Hashemi[71] solved non-linear Lane-Emden type differential equations which are used in the areas of astrophysics. In their work, they tried Gaussian function (GS) as a radial basis function and compare their result with other methods like HFC, Linearization and HPM method. They proved that RBF-DQ method gives better result as compare to other methods.

3.3 RBF-FD Method

3.3.1 Finite Difference (FD) Method

The finite difference method is a new advancement in numerical analysis. FD method is one of the most useful and well known techniques for solving PDEs by approximating derivation of a function with difference equations. In 1911, Richardson[72] proposed the FD approximations for numerically solving PDEs. In this method, the derivatives are approximated by combining neighbouring values of the function with the help of weights. FD method requires a structural grid and weights can be obtained by using local polynomial approximations but lacks geometric flexibility.

3.3.2 Radial Basis Function-Finite Difference (RBF-FD) Method

RBF-FD is one of the hybrid method formed by combining the characteristics of RBF and easy to implement nature of finite difference. The high computational cost of global RBF is the main factor behind the development of RBF-FD method. The accuracy level of the RBF-FD is at par with the global RBF methods. Consider the elliptic partial differential equation discussed in section 3.1. In RBF-FD method, the operator $L\{u(x)\}$ at a point $x = x_i$ is approximated by a linear combination of the function u at n neighbouring points surrounding x_i which constitute its stencil.

$$L\{u(x_i)\} \approx \sum_{j=1}^{n} a_{ij} u(x_j) \qquad (13)$$

where α_{ij} are the weighting coefficients and are determined using interpolation with RBFs unlike FD approximation where we use polynomial. Thus

$$u(x) = \sum_{j=1}^{n} \lambda_j \varphi(r_j) = \sum_{j=1}^{n} \lambda_j \varphi(\varepsilon \left\| x - x_j \right\|) \quad (14)$$

where ϵ is the shape parameter and $\varphi(r_j)$ is some radial function. The weighting function can be obtained by solving the linear system of equations. Substituting equation (14) in equation (13) as

$$L\{\varphi_k(x_i)\} = \sum_{j=1}^{n} \alpha_{ij} \varphi_k(r_j) \text{ , k=1, 2, 3....n} \quad (15)$$

The obtained interpolation matrix becomes sparse which shows very high computational speed and easy to applicable on large scale problems.

Tolstykh[73] first refers RBF-FD approximation in a conference representation. It was then introduced independently by Shu, Ding and Yeo[24] and Liu, Wang[74]. Wright[75] also mentioned RBF-FD in his Ph.D thesis. Tolstykh and Shirobokov[9] construct local support approximation for derivatives in the same context as that of finite difference method. They applied their technique for various types of elasticity equations and find that their results are quite accurate. They considered MQ as RBF for solving the numerical examples.

Chandhini and Sanyasiraju[76] applied RBF-FD to solve convection-diffusion steady type equations. They showed in their work that by changing the shape parameter in MQ-RBF, solution can be highly improved.

Chinchapatnam at el.[77] used the RBF-FD method for incompressible Navier-Stokes equations in stream function. The stencils become one-sided, near boundaries. In order to remove this, a novel ghost node strategy was used i.e. outside the domain, a sheet of nodes is placed which is known as ghost nodes. This strategy is used to overcome the no slip boundary condition.

Wright and Fornberg[78] conceptually improved the method by proposing a compact FD method as RBF-HFD. In scattered FD formulas, symmetries are not available so the accuracy of the formulas cannot be increased. They generalize the concept of compact FD formula and propose this method. They also showed that the RBF-FD and RBF-HFD behave exactly as standard FD and compact HD formulas in the $\epsilon \to 0$ limit (near flat shape parameter), when the stencil nodes are arranged properly.

The stability of RBF-FD method for purely convective PDEs is discussed by Fornberg and Lehto in their work[79]. With the help of traditional hyperviscosity, they develop a filter approach. They also proposed a filter approach for global RBF. The technique also permits the use of larger RBF-FD stencils and therefore much more accurate. Earlier to this, stencils can be around n = 15 nodes whereas with the help of this technique n-values up to 100 were used for higher accuracies. Now we have two main hyperviscosity approaches that is best for global and RBF-FD approximations, one is A^{-1} method for global RBF and other is powers of Laplacian for RBF-FD as discussed in their paper.

The accuracy and computational efficiency for geosciences large scale modelling problems is established by Flyer et al.[80] using RBF-FD. The authors used RBF-FD method for non-linear system of convective PDEs on the sphere and compared to Galerkin model and a spherical harmonic transform model which are already used for these types of equations. They concluded that RBF-FD method require hyperviscosity to stabilize the wide range of time and spatial scales.

Shankar et al.[81] in 2014 developed an RBF-FD to solve diffusion and reaction-diffusion equations on general 1D surfaces embedded in 2D domains. In their study, they simulate the reaction-diffusion equations on stationary platelets that are suspended in blood. The study modified the earlier known method Augmented Forcing Method (AFM) with the help of symmetric RBF Hermite interpolation and eliminated the drawback of AFM like separation constraints on platelets.

Wright et al.[82] improved the above proposed method[81] by designing a numerical method which is more stable. The study shows that their shape parameter optimization approach stabilizes the RBF-FD without hyperviscosity. They also discussed the convergence of their method for diffusion equation $\frac{\partial u}{\partial t} = v\Delta_M u + f(t,u)$, where

$v > 0$ is the diffusion coefficient and $f(t,u)$ is the forcing term on some standard surfaces. They tried two different strategies, one with increasing condition number as the points on the surface increases and other with fixed condition number.

Avazzadeh et al.[83] implement the RBF-FD method for solving a time fractional telegraph equation defined by caputo sense for $1 < \alpha \leq 2$. They first discretize the problem in the time direction using finite difference scheme and then approximate the solution using RBF. Numerical solution of examples shows that the method can be applied to solve different types of fraction PDEs.

Recently Flyer et al.[84] used polyharmonic spline (PHS) as RBF and construct a modified RBF-FD method. They gained high accuracy with good conditioning of interpolation matrix with the help of their method. They also demonstrate that with the help of polynomials with polyharmonic spline one can eliminate the saturation errors.

Alpesh et al.[85] applied RBF-FD method to solve PDE in finance. In their work, the Asian option equation is discretized on time interval by Crank-Nicolson Scheme and option price by RBF based scheme. In the same direction, they solve the one dimensional wave equation with integral condition[86]. As discussed, shape parameter effects the accuracy of the RBF methods so they choose the optimal value of the shape parameter with the help of error and trail basis. The authors effectively compare their result with literature.

3.4 RBF-PU Method

3.4.1 Partition of Unity Method (PUM)

The partition of unity finite element method is proposed by Babuska and Melenk[87] in 1997 and applied the method to solve PDEs. This method partition the region Ω into M gently overlapping subdomains Ω_j Ω_j in such a way that

$$\bigcup_{j=1}^{M} \Omega_j \supseteq \Omega$$

This approach require to choose a family of compactly supported, continuous function w_j such that at every point x in Ω $\sum_{j=1}^{M} w_j(x) = 1$

where w_j is supported on $\overline{\Omega_j}$. For every subdomain Ω_j, let u_j be a local approximation. Then a function f is approximated on each subdomain to form the global approximation for the data on entire domain Ω

$$S_f(x) = \sum_{j=1}^{M} u_j(x) w_j(x), \qquad x \in \Omega$$

3.4.2 Radial Basis Function Partition of Unity Method (RBF-PUM):

In RBF-PUM, local interpolants are constructed on subdomains also called as patches and then combine to form a global approximation using weight functions that form a partition of unity. In this method, RBF is used for local approximation. In 2001, Wendland[31] combines the PUM with the RBF to solve large scale problems for the first time. For mathematical formulation, consider the partial differential equation (2). As discussed the local approximation $u_j(x)$ on overlapping subdomains $\Omega_j, \quad j = 1,2,3,\dots,M$ combine with weight function $w_j, \quad j = 1,2,3,\dots,M$, to form the global approximation $\overline{u}(x)$ as

$$\overline{u}(x) = \sum_{j=1}^{M} u_j(x) w_j(x)$$

The weight functions are from the family of compactly supported continuous function that form the partition of unity. The PDE (2) is discretized with collocation method as

$$L\{\bar{u}(x_k)\} = \sum_{j=1}^{M} L\left(u_j(x_k)w_j(x_k)\right) = f(x_k), \qquad x_k \epsilon\, \Omega, k = 1,2,3, \ldots, N_i$$

$$\bar{u}(x_k) = \sum_{j=1}^{M} u_j(x_k)w_j(x_k) = g(x_k) \qquad x_k \epsilon\, \partial\Omega, k = N_{i+1}, \ldots, N$$

As the local approximation $u_j(x)$ are RBF approximations

$$u_j(x) = \sum_{i=1}^{n_j} \lambda_i^j \varphi_i^j(x) \tag{16}$$

where n_j are the number of nodes that lies in Ω_j, λ_i^j are the unknown and $\varphi_i^j(x) = \phi(\varepsilon\|x - x_i^j\|)$ are the radial basis functions. Local matrix can be formed with the help of the equation (16) using local nodes $x_k, \quad k = 1,2,3 \ldots . n_i \;\; in\; \Omega_j$ $x_k, \quad k = 1,2,3 \ldots . n_i \;\; in\; \Omega_j$. Finally the global matrix L can be formed by assembling the local matrices which results in a system of equation to be solved. The interested reader can refer[87] for further reading. The RBF-PUM provides a way to reduce the computational cost while maintaining high accuracy. The main advantage of the RBF-PUM method is to retain the geometrical flexibility in high dimensional problems, to facilitate adaptive approximation and to overcome computation cost.

Cavoretto and DeRossi[88] proposed algorithm for spherical interpolation for large scale problem using zonal basis function and further proposed the technique using partition of unity method. The authors used spherical radial basis function for local approximation. The method is parallelizable i.e. number of operations can be performed in parallel. Caroretto[89] further extended his work and proposed a partition of unity algorithm which partitions the domain in cells using crossed strips. It is basically a cell based searching procedure. The author extended the above discussed 2D algorithm to 3D using cube partition searching procedure.

Larsson et al.[90] investigated the application of RBF-PUM for the solution of parabolic time dependent PDEs. The authors considered two dimensional convection–diffusion equations arising in financial applications. In this paper, RBF-PUM is compared with already known finite difference and pseudospectral methods and the authors found that RBF-PUM is either more than or as accurate as that of pseudospectral method. The method is in local adaptive nature that can be applied to irregularly shaped domains. The authors proved the stability of the method with the help of theoretical and experimental techniques. Heryudono[91] further improved the RBF-PUM by defining certain preconditioning strategies. The author used generalized minimal residual method to take benefit of the sparsity induced by local approximation and use it as an iterative solver. The matrix generated by RBF-PUM is non symmetric, ill conditioned but sparse so there is a need of effective preconditioner. The authors discussed and compared different preconditioning techniques based on LU factorization and recommend no fill incomplete factorization preconditioner. When shape parameter value is small, RBF-QR method is used to avoid ill conditioning.

4. Concluding Remarks

Over the years, the numerical methods based on RBF grow rapidly due to their meshfree nature. This review presents a insight on the methods which are based on radial basis function formulation. We try to highlight some of the developments that have been taken place in recent years. The methods are presented with mathematical formulation to make the method understandable to the readers. It is hoped that this article familiarizes the reader with the RBF methods. The Kansa collocation method is the simplest meshfree method to solve partial differ-

ential equations. Through the years, the RBF collocation method developed a lot. As the RBF collocation methods results in fully populated matrices which increase the computational cost and fail to perform for large problems so there is always a need to find alternatives to overcome the difficulties. Among all these alternatives, the local collocation RBF methods are popular till date due to their local adaptivity. The local approximation helps to avoid the ill conditioning of the matrices that arises in global approximation. Various hybrid methods were introduced to overcome the difficulties arises in Kansa collocation method. In hybrid methods, the key strength of the RBFs combines with traditional techniques like FD to get the best out of RBFs. These methods help in reduction of computational cost and become very effective in solving large scale problems. For smooth RBFs, the small value of the shape parameter leads to high accuracy but the near flat radial basis ($\epsilon \rightarrow 0$) results in the ill conditioning of the interpolation matrices. Several algorithms that allows stable algorithm for all values of shape parameter were introduced to address this issue. The stable algorithm like RBF-QR works for all values of shape parameter even the values nearest to zero and provides significant improvement in accuracy. The described methods can further be extended by improving the reliability of the RBF methods and by exploring the effect of scaling and node refinement on the accuracy and stability of RBF approximations. Effectiveness and scalability of the RBF methods to solve high dimensional PDEs is still under consideration.

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