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A Survey on Triangular Number, Factorial and Some Associated Numbers

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Abstract

Objectives: The paper aims to present a survey of both time-honored and contemporary studies on triangular number, factorial, relationship between the two, and some other numbers associated with them. **Methods:** The research is expository in nature. It focuses on expositions regarding the triangular number, its multiplicative analog – the factorial and other numbers related to them. **Findings:** Much had been studied about triangular numbers, factorials and other numbers involving sums of triangular numbers or sums of factorials. However, it seems that nobody had explored the properties of the sums of corresponding factorials and triangular numbers. Hence, explorations on these integers, called factoriangular numbers, were conducted. Series of experimental mathematics resulted to the characterization of factoriangular numbers as to its parity, compositeness, number and sum of positive divisors and other minor characteristics. It was also found that every factoriangular number has a runsum representation of length k, the first term of which is $(k-1)! + 1$ and the last term is $(k-1)! + k$. The sequence of factoriangular numbers is a recurring sequence and it has a rational closed-form of exponential generating function. These numbers were also characterized as to when a factoriangular number can be expressed as a sum of two triangular numbers and/or as a sum of two squares. **Application/ Improvement:** The introduction of factoriangular number and expositions on this type of number is a novel contribution to the theory of numbers. Surveys, expositions and explorations on existing studies may continue to be a major undertaking in number theory.

Keywords: Factorial, Factorial-like number, Factoriangular number, Polygonal number, Triangular number

1. Introduction

In the mathematical field, a sense of beauty seems to be almost the only useful drive for discovery[1] and it is imagination and not reasoning that seems to be the moving power for invention in mathematics[2]. It is in number theory that many of the greatest mathematicians in history had tried their hand[3] paving the way for mathematical experimentations, explorations and discoveries. Gauss once said that the theory of numbers is the queen of mathematics and mathematics is the queen of science[4].

The theory of numbers concerns the characteristics of integers and rational numbers beyond the ordinary arithmetic computations. Because of its unquestioned historical importance, this theory had occupied a central position in the world of both ancient and contemporary mathematics.

As far back as ancient Greece, mathematicians were studying number theory. The Pythagoreans were very much interested in the somewhat mythical properties of integers. They initiated the study of perfect numbers, deficient and abundant numbers, amicable numbers, polygonal numbers, and Pythagorean triples. Since then, almost every major civilization had produced number theorists who discovered new and fascinating properties of numbers for nearly every century.

Until today, number theory has shown its irresistible appeal to professional, as well as beginning, mathematicians. One reason for this lies in the basic nature of its problems[5]. Although many of the number theory problems are extremely difficult to solve and remain to be the

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most elusive unsolved problems in mathematics, they can be formulated in terms that are simple enough to arouse the interest and curiosity of even those without much mathematical training.

More than in any part of mathematics, the methods of inquiry in number theory adhere to the scientific approach. Those working on the field must rely to a large extent upon trial and error and their own curiosity, intuition and ingenuity. Rigorous mathematical proofs are often preceded by patient and time-consuming mathematical experimentation or experimental mathematics.

Experimental mathematics refers to an approach of studying mathematics where a field can be effectively studied using advanced computing technology such as computer algebra systems[6]. However, computer system alone is not enough to solve problems; human intuition and insight still play a vital role in successfully leading the mathematical explorer on the path of discovery. And besides, number theorists in the past have done these mathematical experimentations only by hand analysis and computations.

Among the many works in number theory that can be done through experimental mathematics, exploring patterns in integer sequences is one of the most interesting and frequently conducted. It is quite difficult now to count the number of studies on Fibonacci sequence, Lucas sequence, the Pell and associated Pell sequences, and other well-known sequences. Classical number patterns like the triangular and other polygonal and figurate numbers have also been studied from the ancient up to the modern times by mathematicians, professionals and amateurs alike, in almost every part of the world. The multiplicative analog of the triangular number, the factorial, has also a special place in the literature being very useful not only in number theory but also in other mathematical disciplines like combinatorial and mathematical analysis. Quite recently, sequences of integers generated by summing the digits[7] were also being studied.

Integer patterns or sequences can be described by algebraic formulas, recurrences and identities. As an instance, the triangular number (T_k), for $k \geq 1$, is determined by $T_k = k(k+1)/2$. Given T_k the next term in the triangular number sequence is also determined by recurrence: $T_{k+1} = T_k + k + 1$.

Some identities on triangular numbers are also given in the literature, for instance[8-10]: $T_k + T_{k-1} = k^2$; $T_k + T_{k+1} = (k+1)^2$; $T_{k+1}^2 - T_k^2 = (k+1)^3$; $kT_{k+1} = (k+2)T_k$; $T_{k^2} = T_k^2 + T_{k-1}^2$; $T_{2k} = 3T_k + T_{k-1}$; $T_{2k} - 2T_k = k^2$; and $8T_k + 1 = (2k+1)^2 \cdot$

There is also an identity involving the triangular number and factorial. This relationship between factorials and triangular numbers is given by[11,12]: $(2n)! = 2^n \prod_{k=1}^{n} T_{2k-1}$. Aside from this, there is a somewhat natural relation on these factorials and triangular numbers: the triangular number is regarded as the additive analog of factorial.

The current work aims to present a survey of both time-honored and contemporary studies on triangular number and other polygonal numbers, factorial and factorial-like numbers, and some other related or associated numbers. It also includes some interesting results of recent studies conducted by the author regarding the sum of corresponding factorial and triangular number, which is named as *factoriangular number*.

2. Survey on Triangular Number, Factorial and Related Numbers

The history of mathematics in general and the history of number theory in particular are inseparable. Number theory is one of the oldest fields in mathematics and most of the greatest mathematicians contributed for its development[3]. Although it is probable that the ancient Greek mathematicians were largely indebted to the Babylonians and Egyptians for a core of information about the properties of natural numbers, the first rudiments of an actual theory are generally credited to Pythagoras and his followers, the Pythagoreans[5].

2.1 Triangular Number and Other Polygonal Numbers

An important subset of natural numbers in ancient Greece is the set of polygonal numbers. The name polygonal number was introduced by Heysicles to refer to positive integers that are triangular, oblong, square, and the like[13]. These numbers can be considered as the ancient link between number theory and geometry.

The characteristics of these polygonal numbers were studied by Pythagoras and the Pythagoreans. They depicted these numbers as regular arrangements of dots in geometric patterns. The triangular numbers were represented as triangular array of dots, the oblong numbers as rectangular array of dots, and the square numbers as square array of dots. They also found that an oblong number is a sum of even numbers; a triangular number is a sum of positive integers; and a square number is a sum of odd numbers[13]. In modern notation, if the triangular number is denoted by T_k, the oblong number by O_k, and the square number by S_k, then

$$T_k = \sum_{i=1}^{k} i = 1+2+3+...+k \ ;$$

$$O_k = \sum_{i=1}^{k} 2i = 2+4+6+...+2k \text{; and}$$

$$S_k = \sum_{i=1}^{k} (2i-1) = 1+3+5+...+(2k-1) \cdot$$

It is very evident that $2T_k = O_k$. Relating triangular and square numbers, Nicomachus found that $T_k + T_{k+1} = S_{k+1} = (k+1)^2$ or $T_{k-1} + T_k = S_k = k^2$. He also noted that triangular numbers can be produced from square and oblong numbers[13] in particular, $S_k + O_k = T_{2k}$ and $O_k + S_{k+1} = T_{2k+1}$. In addition, Plutarch also found[13] that $8T_k + 1 = S_{2k+1} = (2k+1)^2$.

Further, Nicomachus proved[13] that, if the pentagonal number is denoted by $P^5{}_k$, then for $k>1$, $P^5{}_k = 3T_{k-1} + k$ and $P^5{}_k = S_k + T_{k-1}$.

If the polygonal number is be denoted by $P^n{}_k$, where n is the number of sides, then the triangular number can be denoted by $P^3{}_k$ and the square number by $P^4{}_k$. Using these new notations, the relations among pentagonal, square and triangular numbers is now given, for $k>1$, by $P^5{}_k = P^4{}_k + P^3{}_{k-1}$. Nicomachus generalized this and claimed that, for $k>1$ and $n \geq 3$, $P^n{}_k = P^{n-1}{}_k + P^3{}_{k-1}$ holds for every n-gonal number[14]. This shows that every polygonal number can be generated using the triangular numbers.

For instance, squares can be generated using $S_k = T_k + T_{k-1}$, as Nicomachus observed; pentagonal numbers using $P^5{}_k = S_k + T_{k-1} = T_k + 2T_{k-1}$; hexagonal numbers using $P^6{}_k = P^5{}_k + T_{k-1} = T_k + 3T_{k-1}$; and in general, n-gonal numbers using $P^n{}_k = P^{n-1}{}_k + T_{k-1} = T_k + (n-3)T_{k-1}$, which is used to establish the closed formula for any kth polygonal number[14], for $k \geq 1$ and $n \geq 3$,

$$P^n{}_k = \frac{k(kn-2k-n+4)}{2} \cdot$$

Other established relations between triangular and other polygonal numbers are as follows[10]: $P^6{}_k = T_{2k-1}$ and $P^5{}_k = \frac{1}{3} T_{3k-1}$.

2.2 Contemporary Studies on Triangular Numbers

Triangular numbers, though having a simple definition, are amazingly rich in properties of various kinds, ranging from simple relationships between them and the other polygonal numbers to very complex relationships involving partitions, modular forms and combinatorial properties[8]. Many other important results on these numbers have been discussed in the literature. A theorem of Fermat states that a positive integer can be expressed as a sum of at most three triangular, four square, five pentagonal, or n n-gonal numbers. Gauss proved the triangular case[10,15]. Euler left important results regarding this theorem, which were utilized by Lagrange to prove the case for squares and Jacobi also proved this independently[16]. Cauchy showed the full proof of Fermat's theorem[15,16].

The expression of an integer as a sum of three triangular numbers can be done in more than one way and Dirichlet showed how to derive such number of ways[10]. The modular form theory can also be used to calculate the representations of integers as sums of triangular numbers[17]. The ways a positive integer can be expressed as a sum of two n-sided regular figurate numbers can also be generalized[18]. Furthermore, it was shown that a generating function manipulation and a combinatorial argument can be used on the partitions of an integer into three triangular numbers and into three distinct triangular numbers, respectively[19].

The theory of theta functions can be used also to compute the number of ways a natural number can be expressed as a sum of three squares or of three triangular numbers[20]. Following that, several studies on the mixed sums of triangular and square numbers were conducted.

Any positive integer was shown to be a sum of two triangular numbers and an even square and that each natural number is equal to a triangular number plus $x^2 + y^2$ with x not congruent to y modulo 2 and where x and y are integers[21]. Three conjectures on mixed sums of triangular and square numbers were verified[21], the first for $n \leq 15000$ while the second and third for $n \leq 10000$. The second conjecture was later proven by using Gauss-Legendre Theorem and Jacobi's identity[22]. The first conjecture had been later proven[23] as well while the generalized Riemann hypothesis implied the third conjecture about which explicit natural numbers may be represented[15].

Another conjecture states that for non-negative integers m and n, every sufficiently large positive integer can be expressed as either of the following: (1) $2^m x^2 + 2^n y^2 + T_z$, (2) $2^m x^2 + 2^n T_y + T_z$, (3) $2^m T_x + 2^n T_y + T_z$, (4) $x^2 + 2^n \cdot 3y^2 + T_z$, (5) $x^2 + 2^n \cdot 3T_y + T_z$, (6) $2^n \cdot 3x^2 + 2T_y + T_z$, (7) $2^n \cdot 3T_x + 2T_y + T_z$, (8) $2^n \cdot 5T_x + T_y + T_z$, (9) $2T_x + 3T_y + 4T_z$, (10) $2x^2 + 3y^2 + 2T_z$. Six of the ten forms: (1), (4), (5), (6), (9) and (10), were proven while for the four other forms: (2), (3), (7) and (8), counterexamples were found[15].

2.3 Square Triangular Number and Related Numbers

Square triangular numbers or numbers that are both square and triangular are also well-studied. A square triangular number can be written as n^2 for some n and as $k(k+1)/2$ for some k, and hence, is given by the equation

$$n^2 = \frac{k(k+1)}{2}$$

which is a Diophantine equation for which integer solutions are demanded. Solving this equation by algebraic manipulations leads to $(2k+1)^2 - 2(2n^2) = 1$ and by change of variables $a = 2k+1$ and $b = 2n$ becomes $a^2 - 2b^2 = 1$, which is factorable into $(a+\sqrt{2}b)(a-\sqrt{2}b) = 1$ in finding integer solutions. In a more general equation $a^2 - kb^2 = 1$ where k is a fixed integer, if (a,b) is a solution then $k = (b/2)^2$ is a square triangular number[24].

The sequence of square triangular numbers is {1, 36, 1225, 41616, 1413721, 48024900, 1631432881, 55420693056, …}. The k^{th} square triangular number, which can be denoted by St_k, can be obtained from the recursive formula, $St_k = 34St_{k-1} - St_{k-2} + 2$ for $k \geq 3$ with $St_1 = 1$ and $St_2 = 36$. The non-recursive formula, for $k \geq 1$,

$$St_k = \left[\frac{\left(1+\sqrt{2}\right)^{2k} - \left(1-\sqrt{2}\right)^{2k}}{4\sqrt{2}} \right]^2$$

also gives the k^{th} square triangular number[25].

Square triangular numbers are related to balancing numbers. A balancing number is a positive integer k that makes the equation $1+2+\ldots+(k-1) = (k+1)+(k+2)+\ldots+(k+r)$ true for a positive integer r called balancer[26]. The equation is equivalent to $T_{k-1} = T_{k+r} - T_k$ or $T_{k-1} + T_k = T_{k+r}$, which is also the same as $k^2 = (k+r)((k+r+1)/2$. It is clear that when k^2 is a triangular number, k is a balancing number. Hence, when a square triangular number was found, the square root of such is a balancing number.

Cobalancing numbers are closely related to balancing numbers. A cobalancing number is a positive integer k such that $1+2+\ldots+k = (k+1)+(k+2)+\ldots+(k+r)$ for a positive integer r called the cobalancer[27]. Some properties of square triangular, balancing and cobalancing numbers had been determined[28,29]. Congruences for prime subscripted balancing numbers were established as well[30].

The Pell and associated Pell numbers were also linked with balancing and cobalancing numbers[31]. The concept of balancing numbers was also employed[32] to solve a generalized Pell's equation $y^2 - 5a^2x^2 = 4a^2$.

Balancing and cobalancing numbers were also generalized into arbitrary sequences thereby defining the sequence balancing numbers and sequence cobalancing numbers[33]. These were further generalized into t-balancing numbers[34] and sequence t-balancing numbers[35]. New identities were also established together with (a,b)-type balancing and cobalancing numbers[36]. Further, an analog of balancing number, the multiplying balancing number, was also defined[37].

2.4 Triangular Number, Factorial and Factorial-like Numbers

Probably, the most well-known analogs in number theory are the factorials and the triangular numbers. While a triangular number is the sum of positive integers, a factorial is their product. The factorial of n, denoted by $n!$, gives the number of ways in which n objects can be permuted[12]. It is also the total number of essentially different arrangements using all given n objects of distinct sizes such that each object is sufficiently large to simultaneously contain all previous objects[38].

There are also studies regarding the relationship between factorials and triangular numbers. For instance, there is a natural number m such that $n! = 2^m T$ where T is a product of triangular numbers and the number of factors depends on the parity of n[11].

While there are numbers that are both square and triangular, there are also numbers that are both triangular and factorial. These can be determined by solving the Diophantine

$$n! = \frac{k(k+1)}{2}.$$

Some solutions, (n, k), of which are (1, 1), (3, 3) and (5, 15). Hence, 1, 6 and 120 are numbers that are both triangular and factorial. Tomaszewski in sequence A000142 of OEIS conjectured that these are the only such numbers[38].

Factorial sums, certain type of numbers expressed as sums of factorials, and factorials expressed as sums of other types of numbers have been studied also. The sum-of-factorial function[39] is defined by $S(n) = \sum_{k=1}^{n} k!$, which gives the sequence }1, 3, 9, 33, 153, 873, 5913, 46233, 409113, …} or sequence A007489 in OEIS[38].

There are also square numbers that are sums of distinct factorials[39] like $3^2 = 1! + 2! + 3!$, $5^2 = 1! + 4!$, $11^2 = 1! + 5!$ $12^2 = 4! + 5!$, $27^2 = 1! + 2! + 3! + 6!$, $29^2 = 1! + 5! + 6!$, $71^2 = 1! + 7!$, $72^2 = 4! + 5! + 7!$, $213^2 = 1! + 2! + 3! + 7! + 8!$, $215^2 = 1! + 4! + 5! + 6! + 7! + 8!$, $603^2 = 1! + 2! + 3! + 6! + 9!$, $635^2 = 1! + 4! + 8! + 9!$, and $1183893^2 = 1! + 2! + 3! + 7! + 8! + 9!$; triangular numbers that are sums of distinct factorials such as $T_2 = 1! + 2!$, $T_{17} = 1! + 2! + 3! + 4! + 5!$, $T_{108} = 3! + 5! + 6! + 7!$, $T_{284} = 3! + 4! + 5! + 8!$, $T_{286} = 1! + 6! + 8!$, and $T_{8975} = 5! + 9! + 11!$; and triangular numbers equal to a single factorial, for example $T_1 = 1!$, $T_3 = 3!$ and $T_{15} = 5!$.

Factorials can also be expressed as sums of Fibonacci numbers. All factorials that are sums of at most three Fibonacci numbers had been determined[40] and it was shown[41] also that if k is fixed then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \ldots + m_k!,$$

where F_n is a Fibonacci number, holds for some positive integers m_1, ..., m_k.

Factorial-like numbers also abound the literature. These factorial-like numbers are products of numbers in a sequence other than the consecutive natural numbers from 1 to k. Some of these are the double factorials, the primorials, and the polygorials. The product of even numbers, $2 \cdot 4 \cdot 6 \cdots (2k) = (2k)!!$ and the product of odd numbers, $1 \cdot 3 \cdot 5 \cdots (2k-1) = (2k-1)!!$ are called double factorials[42]. A primorial, which is an analog of the usual factorial for prime numbers, is a product of sequential prime numbers[43]. Polygorials[14] are products of sequential polygonal or n-gonal numbers, the triangular polygorials, square polygorials, pentagonal polygorials or pentagorials, hexagonal polygorials or hexagorials, and so forth. For instance, if $\wp^n{}_k$ denotes the k^{th} n-gorial for $n \geq 3$ and $k \geq 1$, then the k^{th} triangular polygorial is given by $\wp^3{}_k = 1 \cdot 3 \cdot 6 \cdots T_k$ and the k^{th} square polygorial by $\wp^4{}_k = 1 \cdot 4 \cdot 9 \cdots k^2$. $\wp^3{}_k$ is the same as $\prod_{i=2}^{k} T_{k-1}$, which gives the sequence {1, 3, 18, 180, 2700, 56700, …} or sequence A006472 in OEIS[38].

3. Findings and Discussion

Much had been studied about triangular numbers, factorials and other numbers involving sums of triangular numbers or sums of factorials. This section now presents an innovative study about the sums of corresponding factorials and triangular numbers, which are being introduced as factoriangular numbers, and some expositions on these numbers. While the name polygorial is a contraction of polygonal and factorial[14]; factoriangular is a contraction of factorial and triangular. As polygorial does not mean a number that is both polygonal and factorial, a factoriangular number does not also mean a number that is both factorial and triangular.

The natural similarity of factorials and triangular numbers motivated the author to add the corresponding numbers of the two sequences in order to form another sequence. Adding the corresponding factorials and triangular numbers gives the sequence of numbers {2, 5, 12, 34, 135, 741, 5068, 40356, 362925, …}.

Curious enough, the author checked if such sequence is already included in the OEIS. The formulation for this sequence is very simple and it is not surprising to find the sequence in OEIS, which is sequence A101292[38]. But what is quite surprising is that there is very little information about this sequence in OEIS, in particular, and in the literature, in general. Aside from the formula and the list of the first 20 numbers in the sequence, provided only are the *Maple* and the *Mathematica* programs for generating the sequence. There are no references indicated and no comments from OEIS contributors and number theory practitioners, despite the fact that the sequence is there in OEIS for some years already. It seems that nobody has yet explored the properties of integers that are sums of two of the most important, most popular, and mostly studied numbers: the factorials and the triangular numbers.

Series of experimental mathematics done by the researcher resulted to the characterization of factoriangular numbers as to its parity, compositeness, number and sum of positive divisors and other minor characteristics[44]. For a positive integer k, the k^{th} factoriangular number, denoted by Ft_k, is given by the formula $Ft_k = k! + T_k$, where $k! = 1 \cdot 2 \cdot 3 \cdots k$ and $T_k = 1 + 2 + 3 + \ldots + k$. Ft_k is even if $k = 1$ or if k is of the form $4r$ (for integer $r \geq 1$) or $4r + 3$ (for integer $r \geq 0$); but it is odd if k is of the form $4r + 1$ (for integer $r \geq 1$) or $4r + 2$ (for integer $r \geq 0$). For $k = 1, 2$, Ft_k is prime; but for $k \geq 3$, Ft_k is composite and it is divisible by k if k is odd and by $k/2$ if k is even. Further, for even k, there is an integer r_1 such that $Ft_k = r_1 + Ft_{k+1}/(k+1)$ and this r_1 is equal to $(k^2-2)/2$; while for odd k, there is an integer r_2 such that $2Ft_k = r_2 + 2Ft_{k+1}/(k+1)$ and this r_2 is equal to $k^2 - 2$.

In another paper of the author, an exposition on the runsum representations of factoriangular numbers was done[45]. A runsum is a sum of a run of consecutive positive integers[46]. In determining the runsum representations of Ft_k, the discussion on rumsums of length k given in another article[47] served as the groundwork. It was found out that every Ft_k has a runsum representation of length k, the first term of which is $(k-1)!\ +\ 1$ and the last term is $(k-1)!\ +\ k$ and these first terms and last terms form sequences similar to A038507 and A213169 of OEIS[38], respectively. Further, for $k \geq 2$, the sums of the first and last terms of the runsums of length k of Ft_k form another interesting sequence with the formula, $2(k-1)!+k+1$, which is equal to twice the Ft_k divided by k. In addition, two identities were established: $Ft_k = T_{(k-1)!+k} - T_{(k-1)!}$ and $k! = T_{(k-1)!+k} - T_{(k-1)!} - T_k$, where T_i is the i^{th} triangular number.

Consequently, it was found that the sequence of factoriangular numbers is a recurring sequence with a rational closed-form exponential generating function[48]. These numbers follow the recurrence relations

$$Ft_{k+1} = (k+1)\left(Ft_k - \frac{k^2-2}{2}\right), \quad \text{for} \quad k \geq 1, \quad \text{and}$$

$$Ft_k = k\left(Ft_{k-1} - \frac{k^2-2k-1}{2}\right), \text{ for } k \geq 2.$$

And the exponential generating function of the sequence of Ft_k is given by the formula:

$$E(x) = \frac{2+(2-5x^2+2x^3+x^4)e^x}{2(1-x)^2}, \ -1 < x < 1.$$

Moreover, interesting expositions on factoriangular numbers expressed as sum of two triangular numbers and/or as sum of two squares were also conducted[49]. It was found that for natural numbers k, r, and s and i^{th} triangular number T_i, the following were established: (1) only two solutions, $(Ft_k, T_k) = (Ft_1, T_1), (Ft_3, T_3)$, satisfy the relation $Ft_k = 2T_k$; (2) $Ft_k = 2T_r$ if and only if $4Ft_k + 1$ is a square; (3) $Ft_k = T_r + T_k$ if and only if $8k!\ +\ 1$ is a square; and (4) $Ft_k = T_r + T_s$ if and only if $8Ft_k + 2$ is a sum of two squares.

4. Conclusion

The different types of numbers were among the foremost subjects of study in the oldest times. Nevertheless, such study on numbers is not yet complete and most probably, will never be completed. New and fascinating groups or sequences of numbers may be discovered in contemporary times and in the future by both expert number theorists and novice number enthusiasts.

Experimental mathematics is a key tool in discovering new kinds of numbers. However, ingenuity and creativeness still play a vital role. Simple additions or multiplications of existing numbers may lead the investigator to create or invent a new category of numbers.

Surveys, expositions and explorations on existing studies may continue to be a major undertaking in number theory and in mathematics in general. Conjectures need to be proven or otherwise and open questions need to be answered.

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