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Generalized Fuzzy σ - Algebra and Generalized Fuzzy Measure on Soft Sets

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Abstract

In this paper, we define soft fuzzy De Morgan algebra, soft generalized fuzzy σ algebra and soft generalized fuzzy measure. Related to these definitions, we give some examples, lemmas and interesting results on soft generalized fuzzy measure space.

Keywords: Soft Fuzzy De Morgan Algebra, Soft Fuzzy Measure, Soft Fuzzy σ Algebra

1. Introduction

Dealing with uncertainties is a major problem in many areas such as engineering, medical science, environmental science, social science et al. These kinds of problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these kinds of difficulties, Molodtsov[1] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Maji, Biswas and Ray[2] studied the theory of soft sets and developed several basic nations of soft sets theory in 2003.

Later, Nakajima[3] defined measurability on ring of generalized fuzzy sets in his study by generalizing L - fuzzy sets and gave some results related to this measurability. In another study Nakajima[4] generalized a complete Heyting algebra on fuzzy sets family and obtained several results. E. P. Klement[5] on the other hand, analyzed the relationship between fuzzy σ - algebra and classical σ - algebra. Klement and Schywla[6] generalized fuzzy σ - algebra by comparing classical measure and fuzzy measure. By defining soft σ - algebra on X universal sets and E parameters sets, Khameneh and Kılıçman[7] analyzed some characteristics of this algebra. Akram et al.[8] introduced the concept of bipolar fuzzy soft Γ-subsemigroup and bipolar fuzzy soft Γ-ideals in a Γ-semigroup. It is proved that the extended union, extended intersection, restricted union and restricted intersection of two same kind bipolar fuzzy soft Γ-ideals over a Γ-semigroup produced a same kind's bipolar fuzzy soft Γ-ideal. Malik et al.[9] shown that every fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal is a fuzzy (weak, strong, reflexive) hyper BCK-ideal. Yaqoob et al.[10] defined concept of interval valued intuitionistic fuzzy ternary subsemigroup (ideal) of a ternary semigroup with respect to interval t-norm T and interval t-conorm S and the characteristic properties are described.

Sahin[11] defined generalized σ - algebra on subsets of ring of fuzzy sets GF(X) by generalizing some measure concepts. Yinsheng[12] analyzed some characteristics of fuzzy measures by defining De Morgan algebra. Using the definition of De Morgan algebra Esteva[13] defined fuzzy De Morgan algebra and pointed out that De Morgan algebra and fuzzy De Morgan algebra were isometric. Sahin[11] generalized the proofs of Lemma 2.2 and Lemma 2.3 given by Nakajima[4]. In this study, Lemma 2.2 and Lemma 2.3 and their generalized proofs were defined in soft sets including soft fuzzy De Morgan algebra, soft fuzzy measure and soft fuzzy σ - algebra.

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2. Preliminaries

2.1 Definition:

A family GF (X), which is closed under operations ∨ and ∧, is called a ring of generalized fuzzy subsets of X if the following conditions are satisfied[3]:

- GF (X) is a complete Heyting algebra with respect to ∨ and ∧
- GF (X) contains P(X), the power set of X as a sublattice of GF (X)
- The operations ∨ and ∧ coincide with the set operations ∪ (union) and ∩(intersection), respectively, in P(X)
- For any element A in GF(X), A ∨ X=X and A ∧ ∅ = ∅

2.2 Definition: A family S ⊂ GF (X) is called a generalized fuzzy σ-algebra of GF (X), if the following conditions are satisfied[11]:

- ∅ ∈ S and X ∈ S
- for A ∈ S, A ∧ A^c = 0, A ∨ A^c = 1, when$(A^c)^c$ = A, A ∈ S
- $\{A_i\}_{i\in N} \subset S \Rightarrow \bigcup_{i=1}^{\infty} A_i \in S.$

2.3 Definition: A function $m : S \to R_+^* = R_+ \cup \{+\infty\}$ is called a generalized fuzzy measure on S if the following conditions are satisfied[11]:

- $for\ A \in S,\ m(A) \geq 0,\ m(\varnothing) = 0,$
- $for\ A, B \in S,\ if\ m(A) \leq m(B)$ then $A \leq B,$
- $for\ A, B \in S,\quad m(A \vee B) + m(A \wedge B) = m(A) + m(B)$
- $(A_n)_{n\in N} \subset S^N, A \in S : (A_n) \uparrow A \Rightarrow m(A_n) \uparrow m(A)$

The pair (X,S) is called a generalized fuzzy measurable space.

2.4 Definition: A pair (F, A) is called a soft set over U, if F is a mapping given by F : A → P(U)where U refers to an initial universe, E is a set of parameters, P (U) is the power set of U and A ⊆ E, that is , a soft set over U is a parameterized family of subsets of the universe[1] U.

2.5 Definition: Let[14] U be an initial universe set and E be a set of parameters. Let P (U) denote the set of all fuzzy sets in U. Then (F, A) is called a fuzzy soft set over U where A ⊆ E and F is a mapping given by F : A → P(U).

In general, ε ∈ A, F(ε) is a fuzzy set in U and it is called a fuzzy value set of parameter ε. If for every ε ∈ A, F(ε) is a crisp subset of U, and then (F, A) is degenerated to be standard soft set. Thus, from the above definition, it is clear that fuzzy soft sets are a generalization of standard soft sets.

2.6 Definition: A triple M = (f, U, L), where L is a complete lattice, f : U → L is a mapping and U is a universe set is called the soft lattice[15].

2.7 Definition: Let[16] f_L be a fuzzy soft set over U, ∧ and ∨ be two binary operations on f_L. If the elements of f_L are equipped with two commutative and associative binary operations ∧ and ∨ which are connected by the absorption law, then the algebraic structure (f_L, ∧, ∨) is called a fuzzy soft lattice.

2.8 Definition: Let[16] (f_L, ∧, ∨, ≤) be a fuzzy soft lattice. Then f_L is called distributive fuzzy soft lattice if

$$f_L(u_1) \wedge (f_L(u_2) \vee f_L(u_3)) = (f_L(u_1) \wedge f_L(u_2)) \vee (f_L(u_1) \wedge f_L(u_3))$$
$$f_L(u_1) \vee (f_L(u_2) \wedge f_L(u_3)) = (f_L(u_1) \vee f_L(u_2)) \wedge (f_L(u_1 \vee) f_L(u_3))$$

2.9 Definition: Let[16] (f_L, ∧, ∨, ≤) be a fuzzy soft lattice. If every subset of f_L have both a greatest lower bound and a least upper bound, then it is called complete fuzzy soft lattice.

2.10 Definition: Let[16] (f_L, ∧, ∨, ≤) be a fuzzy soft lattice and f_M ⊆ f_L If f_M is a fuzzy soft lattice with the operations of f_L and then f_M is called a fuzzy soft sublattice of f_L.

2.11 Definition: A collection m of soft sets over X is said to be a soft σ- algebra on X if m has the following properties[7]:

- $\tilde{X} \in m$
- $If\ (F, E) \in m\ then\ (F, E)^r \in m$
- It is closed under countable union.
- If m is a soft σ-algebra in X, then (X, m, E) is called a soft measurable space and the members of m are called the soft measurable sets in X.

2.12 Definition: Let[17] (f, A) be a non-null fuzzy soft set over a ring R . Then (f, A) is called a fuzzy soft ring over R if and only if f(a) = f_a is a fuzzy subring of R for each a ∈ A, that is

$(FSR1) f_a(x+y) \geq T\{f_a(x), f_a(y)\}$

$(FSR2) f_a(-x) \geq f_a(x)$

$(FSR3) f_a(x.y) \geq T\{f_a(x), f_a(y)\}$ for all $x, y \in R$.

2.13 Definition: Let[12] M whose the least element is 0 and the greatest element is I, be a distributive lattice and c be a complement in M such that if

- $0^c = I$, $I^c = 0$
- $(a^c)^c = a$
- $(a_1 \vee a_2)^c = a_1^c \wedge a_2^c, (a_1 \wedge a_2)^c = a_1^c \vee a_2^c$

Then (M, c) is called De Morgan algebra.

2.14 Definition: Let[12] X be a universe of discourse. We denote the lattice of fuzzy subsets of X with values on [0, 1] by L (x) (P(U), ∩, ∪), ∩ and ∪ , the usual max and min. operations and P(X) $[0, 1]^X$ It is known that L(X) is a complete, infinitely distributive lattice with $\max X\{X(x)=1 \text{ for any } x \in X\}$, and $\min \varnothing\{\varnothing(x)=0 \text{ for any } x \in X\}$ and also the Boolean elements of L(X) constitute the Boolean algebra P(X) of crisp subsets of X.

3. Generalized Fuzzy σ - Algebra and Generalized Fuzzy Measure on Soft Sets

Firstly, definitions of soft fuzzy σ - algebra and soft fuzzy measure are given and then a theorem is proved using some lemmas and their proofs that we defined on soft sets.

3.1 Definition: Let f_M whose, the least element is $\bar{0}$ and the greatest element is $\bar{I}$ be a distributive fuzzy soft lattice and c is the complement in f_M such that if:

- $\bar{0}^c = \bar{I}$, $\bar{I}^c = \bar{0}$
- $((F,A)^c)^c = (F,A)$
  $((F,A) \tilde{\vee} (F,B))^c = (F,A)^c \tilde{\wedge} (F,B)^c$
- $((F,A) \tilde{\wedge} (F,B))^c = (F,A)^c \tilde{\vee} (F,B)^c$

Then (f_M, c) is called a soft fuzzy De Morgan algebra.

3.2 Example: For $f_L = [0,1] \tilde{\subseteq} f_L$ interval, let the operation $\tilde{a}^c = \bar{1} - \tilde{a}$ be the complement operation. Similarly, when $\tilde{M} \in R(E)$, $\tilde{M} \tilde{>} \bar{0}$ for $f_L = [0, \tilde{M}]$ interval, $\tilde{a}^c = \tilde{M} - \tilde{a}$, $\tilde{a} \in f_L$ is a complement operation, (f_L, c) is a soft fuzzy De Morgan algebra.

3.3 Example: Let $B \neq 0$ be a set. Let $f_L = 2^B$. For A ∈ f_L, let A^c = B\A operation be the complement operation. Hence the pair (2^B , c) is called a soft fuzzy De Morgan algebra.

Let $f_L^{\tilde{u}}$ be complete operation family from U to L. Given "$\tilde{\leq}$" in $f_L^{\tilde{u}}$ in. For ∀(F, A) ∈ U if $f((F,A)) \tilde{\leq} g((F,A))$, $f \tilde{\leq} g$.

From here in this paper, f_L indicates the soft fuzzy De Morgan algebra and $(f_L^{\tilde{u}}, \tilde{\leq})$ is a soft fuzzy lattice.

3.4 Definition: Let the defined complement operation in $f_L^{\tilde{u}}$ be as follows:

$$(f^c)((F,A)) = f((F,A))^c \, \forall (F,A) \in U$$

Therefore the elements of $f_L^{\tilde{u}}$ are L- soft fuzzy sets.

3.5 Lemma

The pair (f_L,c) is called a soft fuzzy De Morgan algebra.

Proof:

- Let f, g ∈ f_L and f EMBED Equation. DSMT4g. Then $f((F,A)) \tilde{\leq} g((F,A)), \forall (F,A) \in U$.

Thus $f((F,A))^c \tilde{\geq} g((F,A))^c$, and as a result f^c EMBED Equation.

DSMT4g.

- Let f ∈ f_L.

$$(f^c)((F,A)) = f((F,A))^c$$

And

$$(f^c)^c((F,A)) = (f((F,A))^c)^c = f(F,A) \forall (F,A) \in U.$$

As

$$(f^c)^c = f,$$

(f EMBED Equation. DSMT4g)^c(F, A)= ((f(F,A))) EMBED Equation. DSMT4g((g(F, A)))^c = f^c(F, A)
and then

3.6 Definition: A family $H \tilde{\subset} SGF(U)$ is called a soft fuzzy σ-algebra of SGF (U), if the following conditions are satisfied:

- $\varnothing \in$ H and U $\in$ H,

For $(F,A) \in H,\ (F,A)\tilde{\vee}(F,A)^c = \overline{1},\ \ (F,A)\tilde{\wedge}(F,A)^c = \overline{0},$

where $\left((F,A)^c\right)^c = (F,A)$ and $(F,A)^c \in H$,

- $\{(F_i,A)\}_{i\in\mathbb{N}} \tilde{\subset} H \ \Rightarrow \bigcup_{i=1}^{\infty}(F_i,A) \in H$

Since

$f_L = \{f_L^{\tilde{u}} : \tilde{u} \in U\},\ f_L^{\tilde{u}}$ is a complete soft Heyting algebra. For $\tilde{u} \in U$ and $\forall (F,A) \in H \tilde{\subset} SGF(U)$ we have $\mu_{\tilde{A}}(\tilde{u}) \in f_L(\tilde{u})$. In other words, (F, A) is a L - soft fuzzy set ($(F,A) = \tilde{A}$).

3.7 Definition: Let, $(F_1,A),(F_2,A),... \in H \tilde{\subset} SGF(U)$.

If $\forall \tilde{u} \in U, \mu_{\tilde{A}_n}(\tilde{u}) \tilde{\leq} \mu_{\tilde{A}_{n+1}}(\tilde{u})$, then

$\tilde{\vee}_n \mu_{\tilde{A}_n}(\tilde{u}) = \mu_{\tilde{A}}(\tilde{u}) \ \Rightarrow \tilde{A}_n \uparrow A$.

3.8 Definition: $(F,A)\tilde{\wedge}\varnothing = \varnothing,\ (F,A)\tilde{\vee}U = U$ and $H \tilde{\subset} SGF(U)$ are closed under operations $\tilde{\wedge}$ and $\tilde{\vee}$. A function

$m : \mathrm{H} \rightarrow R_+^*(E) = R_+(E) \cup \{+\infty\}$

is called a soft generalized fuzzy measure on H if the following conditions are satisfied:

- for $(F,A) \in H; m((F,A)) \tilde{\geq} \overline{0},\ m(\varnothing) = \varnothing$
- for $(F,A),(F,B) \in H,$ *if* $m((F,A)) \tilde{\leq} m((F,B)) \in H,\ (F,A) \tilde{\leq} (F,B)$
- for $(F,A),(F,B) \in H,$

$$m((F,A)\tilde{\vee}(F,B)) + m((F,A)\tilde{\wedge}(F,B)) = m((F,A)+(F,B))$$

$$\forall ((F_n,A))_{n\in N} \tilde{\subset} H^N, (F,A) \in H : (F_n,A) \uparrow (F,A) \ \Rightarrow \ m((F_n,A)) \uparrow m((F,A))$$

The triple(U, H, E) is called a soft generalized fuzzy measurable space.

3.9 Lemma

Let H = P(U) . Then a function $m : \mathrm{H} \rightarrow R_+^*(E)$ defined as

$$m((F,A)) = \begin{cases} \text{Number of elements of } (F,A), & \text{if } (F,A) \text{ is finite} \\ \infty & \text{, if } (F,A) \text{ is infinite} \end{cases}$$

is a soft generalized fuzzy measure.

Proof: Write $\tilde{\bigcap}$ and $\tilde{\bigcup}$ in place of $\tilde{\wedge}$ and $\tilde{\vee}$, respectively.

- $m(F,A) \tilde{\geq} \overline{0}$ *for* $(F,A) \in P(U) m(\varnothing) = \overline{0}$ *and* $\varnothing \in P(U)$
- For $(F,A),\ (F,B) \in H = P(U),$ *if* $(F,A) \tilde{\subseteq} (F,A),$

then the number of element of (F,A) cannot exceed that of (F, B). Thus $m(F,A) \tilde{\leq} m(F,B)$.

- Let (F,A) (F,B) $\in$ H. Then,

$$m((F,A)\tilde{\vee}(F,B)) + m((F,A)\tilde{\wedge}(F,B)) = m((F,A)) + m((F,B))$$

$$(F,A)\tilde{\vee}(F,B) = (F,A)\tilde{\cup}(F,B)$$

$$(F,A)\tilde{\wedge}(F,B) = (F,A)\tilde{\cap}(F,B)$$

and m' is reduced the two classical measure.

$$(F,A)-(F,B) = (F,A)-((F,A)\tilde{\cup}(F,B))$$

$$(F,A)-(F,B) = ((F,A)\tilde{\cup}(F,B))-(F,A)$$

$$\Rightarrow (F,A)-((F,A)\tilde{\cap}(F,B)) = ((F,A)\tilde{\cup}(F,B))-(F,B). \quad ... \quad (1)$$

Since $m((F,A)-(F,B)) = m((F,A)) - m((F,B))$ *for* $(F,A),\ (F,B) \in H$

for, then

$$m((F,A)) - m((F,A)\tilde{\cap}(F,B)) = m((F,A)\tilde{\cup}(F,B)) - m((F,B)) \quad ... \quad (2)$$

Let $(F,B) \tilde{\subset} (F,A)$. Then from (2), we get,

$(F,A)\tilde{\cap}(F,B) = (F,B),\ (F,A)\tilde{\cup}(F,B) = (F,A)$.

Thus, we have

$$m((F,A)\tilde{\vee}(F,B)) + m((F,A)\tilde{\wedge}(F,B)) = m((F,A)) + m((F,\mathrm{B}))$$

Since $(F,A) \in H\,(F_n,A) \uparrow (F,A) \Rightarrow \lim_{n\to\infty}(F_n,A) = (F,A)$

that is

$$\tilde{\bigcup_{n=1}^{\infty}}(F_n,A) = (F,A)$$

$$\Rightarrow m\left(\lim_{n\to\infty}(F_n,A)\right) = m((F,A)) \Leftrightarrow m((F_n,A)) \uparrow m((F,A)).$$

Define $\mu_{\tilde{A}}^{\circ}$ as $\mu_{\tilde{A}}(\varepsilon) = \{(\varepsilon,\alpha_\varepsilon) \in U \times f_L : \alpha_\varepsilon \tilde{<} \mu_{\tilde{A}}(\varepsilon)\}$. Then, $\mu_{\tilde{A}}^{\circ}$ denotes the area under $\mu_{\tilde{A}}(\varepsilon)$. For the operations $\tilde{\wedge},\ \tilde{\vee}$ used in soft lattice, $f_L(\varepsilon),\ \mu_{\tilde{A}}(\varepsilon)\tilde{\wedge}\mu_{\tilde{B}}(\varepsilon)$ and $\mu_{\tilde{A}}(\varepsilon)\tilde{\vee}\mu_{\tilde{B}}(\varepsilon)$ have their usual meaning. If $\tilde{\wedge},\ \tilde{\vee}$ are operators used in $SGF(U)$, then

$(F,A)\tilde{\wedge}(F,B)$ means $\mu_{\tilde{A}\tilde{\cap}\tilde{B}}$, where $\tilde{\cap}, \tilde{\cup}$, are used in U x L. For $(F,A),\ (F,B)\in H$ consider the following:

$$\mu_{\tilde{A}\tilde{\cap}\tilde{B}} = \{(\varepsilon,\alpha_\varepsilon):\ \alpha_\varepsilon \tilde{<} \mu_{\tilde{A}\cap\tilde{B}}\}$$
$$= \{(\varepsilon,\alpha_\varepsilon):\ \alpha_\varepsilon \tilde{<} \mu_{\tilde{A}\cap\tilde{B}}(\varepsilon) \tilde{\le} \mu_{\tilde{A}}(\varepsilon) \tilde{\le} \mu_{\tilde{B}}(\varepsilon),\ \alpha_\varepsilon \tilde{<} \mu_{\tilde{A}\tilde{\wedge}\tilde{B}}(\varepsilon) \tilde{\le} \mu_{\tilde{B}}(\varepsilon)\}.$$

Then,

$$\alpha_\varepsilon \tilde{<} \mu_{\tilde{A}}(\varepsilon)\ and\ \alpha_\varepsilon \tilde{<} \mu_{\tilde{B}}(\varepsilon) \Rightarrow \alpha_\varepsilon \tilde{<} \min\{\mu_{\tilde{A}}(\varepsilon), \mu_{\tilde{B}}(\varepsilon)\}$$

Thus,

$\mu_{\tilde{A}}(\varepsilon)\tilde{\wedge}\mu_{\tilde{B}}(\varepsilon)$ is valid for $f_L(\varepsilon)$. Here,

$\mu^\circ_{\tilde{A}},\ \mu^\circ_{\tilde{B}} \in U\times L.\ (L=\{f_L(\varepsilon):\ \varepsilon\in U\})$

Since

$$(\varepsilon,\alpha_\varepsilon)\in(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ \Rightarrow (\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{A}},\ (\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{B}}$$

Where

$$(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})(\varepsilon) = \min\{\mu_{\tilde{A}}(\varepsilon), \mu_{\tilde{B}}(\varepsilon)\}$$
$$(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ \tilde{\subset} \mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}},\ \mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}} \tilde{\subset} U\times L,$$

We have

$$(\varepsilon,\alpha_\varepsilon)\in(\mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}}) \Rightarrow \alpha_\varepsilon \tilde{<} \mu^\circ_{\tilde{A}}(\varepsilon)\ and\ \alpha_\varepsilon \tilde{<} \mu^\circ_{\tilde{B}}(\varepsilon),$$
$$\alpha_\varepsilon \tilde{<} \min(\mu_{\tilde{A}}(\varepsilon), \mu_{\tilde{B}}(\varepsilon)) \Rightarrow (\varepsilon,\alpha_\varepsilon)\in(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}}),$$
$$(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ = \mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}}$$

We get

$$(\varepsilon,\alpha_\varepsilon)\in(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})(\varepsilon)$$
$$\Rightarrow \alpha_\varepsilon \tilde{<} (\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})(\varepsilon) = \max\{\mu_{\tilde{A}}(\varepsilon), \mu_{\tilde{B}}(\varepsilon)\}.$$

If $(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})(\varepsilon)=\mu_{\tilde{B}}(\varepsilon)$, then $\alpha_\varepsilon \tilde{<} \mu_{\tilde{B}}(\varepsilon), (\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{B}}$, and hence $(\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{A}}\tilde{\cup}\mu^\circ_{\tilde{B}}$. Similarly, if $(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})(\varepsilon)=\mu_{\tilde{A}}(\varepsilon) \Rightarrow (\varepsilon,\alpha_\varepsilon)\in(\mu^\circ_{\tilde{A}}\tilde{\cup}\mu^\circ_{\tilde{B}})$,

Then

$$(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}}) \tilde{\subset} \mu^\circ_{\tilde{A}}\tilde{\vee}\mu^\circ_{\tilde{B}} \in U\times L \Rightarrow \mu_{\tilde{A}\tilde{\vee}\tilde{B}} \tilde{\subset} (\mu^\circ_{\tilde{A}}\tilde{\cup}\mu^\circ_{\tilde{B}}).$$

Conversely, we get

$$\forall(\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{A}}\tilde{\cup}\mu^\circ_{\tilde{B}} \Rightarrow (\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{A}}\ or\ (\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{B}}.$$

For example, let $(\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{A}}$. then,

$$\alpha_\varepsilon \tilde{<} \mu_{\tilde{A}}(\varepsilon) \Rightarrow \alpha_\varepsilon \tilde{<} \max\{\mu_{\tilde{A}}(\varepsilon), \mu_{\tilde{B}}(\varepsilon)\}$$
$$\Rightarrow \alpha_\varepsilon \in (\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})(\varepsilon)$$
$$\Rightarrow (\varepsilon,\alpha_\varepsilon)\in(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})^\circ$$
$$\Rightarrow (\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}}) = (\mu_{\tilde{A}})^\circ \tilde{\cup} (\mu_{\tilde{B}})^\circ.$$

3.10 Theorem

Let $(\mathrm{H}\in SGF(U))$ be a soft fuzzy σ-algebra of soft generalized fuzzy sets. Let A(H) be a classical σ-algebra determined by $\{\mu^\circ_{\tilde{A}} : \tilde{A}\in\mathrm{H}\}$ on U x L. If $\varphi:(U\times L, A(\mathrm{H}))\to R^*_+(E)$ is a (finite) soft fuzzy measure where $m:\mathrm{H}\to R^*_+(E)$ such that $m(\mu_{\tilde{A}})=\varphi(\mu_{\tilde{A}})$ then m is a soft generalized fuzzy measure.

Proof:

- We have $m(\varnothing)=\varphi(\mu^\circ_\varnothing)=\overline{0}$, where

$$\mu^\circ_\varnothing = \{(\varepsilon,\alpha_\varepsilon)\in U\times f_L : \alpha_\varepsilon \tilde{<} \mu_\varnothing(\varepsilon)=\overline{0}\}.$$

Since no such $\alpha_\varepsilon\in f_L$ can be found, $\mu^\circ_\varnothing$ is a $\varnothing$ set. As φ is a measure, $\varphi(\mu^\circ_\varnothing)=\varphi(\varnothing)=\overline{0}$. Therefore, $m(\mu_\varnothing)=\varphi(\mu^\circ_\varnothing)$.

- Suppose $\mu_{\tilde{A}},\mu_{\tilde{B}}\in\mathrm{H}$. Let $\mu_{\tilde{A}} \tilde{\le} \mu_{\tilde{A}}$. Then $\mu^\circ_{\tilde{A}} = \{(\varepsilon,\alpha_\varepsilon)\in U\times f_L : \alpha_\varepsilon < \mu_{\tilde{A}}(\varepsilon)\}$ and

$$\mu_{\tilde{A}}(\varepsilon) \tilde{\le} \mu_{\tilde{B}}(\varepsilon) \Rightarrow \alpha_\varepsilon \tilde{<} \mu_{\tilde{B}}(\varepsilon).$$

Thus $(\varepsilon,\alpha_\varepsilon)\in\mu^\circ_{\tilde{B}}$. Since φ is a measure, we get

$$\varphi(\mu^\circ_{\tilde{A}}) \tilde{\le} \varphi(\mu^\circ_{\tilde{B}}), \mathrm{m}(\mu^\circ_{\tilde{B}}) = \varphi(\mu^\circ_{\tilde{A}}) \tilde{\le} \varphi(\mu^\circ_{\tilde{B}}) = \mathrm{m}(\mu^\circ_{\tilde{B}}) \Rightarrow \mathrm{m}(\mu^\circ_{\tilde{A}}) \tilde{\le} \mathrm{m}(\mu^\circ_{\tilde{B}}).$$

- For sets $(F,A),(F,B)\in\mathrm{H}$, consider $\mu_{\tilde{A}},\mu_{\tilde{B}}\in H$. Then,

$$m(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}}) + m(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}}) = \varphi((\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ) + \varphi((\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})^\circ) \quad (*)$$

Since, $(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ = \mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}}$ and $(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})^\circ = \mu^\circ_{\tilde{A}}\tilde{\cup}\mu^\circ_{\tilde{B}}$, from **(*)** we have

$$\varphi((\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ) + \varphi((\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}})^\circ) = \varphi((\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}})^\circ) + \varphi(\mu^\circ_{\tilde{A}}) + \varphi(\mu^\circ_{\tilde{B}}) - \varphi(\mu^\circ_{\tilde{A}}\tilde{\cap}\mu^\circ_{\tilde{B}})$$
$$= \varphi(\mu^\circ_{\tilde{A}}) + \varphi(\mu^\circ_{\tilde{B}})$$
$$= m(\mu_{\tilde{A}}) + m(\mu_{\tilde{B}}).$$

Hence,

$$m(\mu_{\tilde{A}}\tilde{\vee}\mu_{\tilde{B}}) + m(\mu_{\tilde{A}}\tilde{\wedge}\mu_{\tilde{B}}) = m(\mu_{\tilde{A}}) + m(\mu_{\tilde{B}}).$$

Let $m(\mu_{\tilde{A}_n})_{n\in N}\in\mathrm{H}^N$ such that $(\mu_{\tilde{A}_n})\uparrow\mu_{\tilde{A}}$. This implies that $\left(\mu^{\circ}_{\tilde{A}_n}\right) \uparrow \mu^{\circ}_{\tilde{A}}$.

Since φ is a measure, $\varphi\left(\mu^{\circ}_{\tilde{A}_n}\right) \uparrow \varphi\left(\mu^{\circ}_{\tilde{A}}\right)$ is true. It follows that

$$m\left(\mu_{\tilde{A}_n}\right) = \varphi\left(\mu^{\circ}_{\tilde{A}_n}\right) \uparrow \varphi\left(\mu^{\circ}_{\tilde{A}}\right) = m\left(\mu_{\tilde{A}}\right).$$

3.11 Definition: If f_M is a sublattice of a lattice f_L, then f_L is said to be an extension of f_M. If f_{L1}, a sublattice of f_L, is the minimal sublattice which contains the sublattice f_M of f_L and the subset $\tilde{K}$ of f_L, then f_{L1} is the extension lattice of f_M obtained by adjoining $\tilde{K}$ and is denoted by $f_{L_1} = f_M\left(\tilde{K}\right)$.

3.12 Lemma

Let f_L be a distributive lattice with the least element $\bar{0}$ and the greatest element $\bar{1}$, and let f_M and $\tilde{K}$ be sub lattices of f_L such that $\max f_M = \max\left(\tilde{K}\right)$ and $\min f_M = \min\left(\tilde{K}\right) = \bar{0}$. Then $f_M\left(\tilde{K}\right)$ is equal to:

$$f_{L_0} = \left\{ \tilde{\bigvee}_{i=1}^{n} \left(\tilde{a}_i \wedge \tilde{x}_i\right) : \tilde{a}_i \in f_M, \tilde{x}_i \in \tilde{K}; n = 1, 2, \ldots \right\}.$$

Proof: An element of f_{L0} can be denoted by $\tilde{\bigvee}_{i=1}^{n}\left(\tilde{a}_i \wedge \tilde{x}_i\right)$ with $\tilde{a}_i$ not being zero for finite number of i's. For $y = \tilde{\bigvee}_{i=1}^{n}\left(\tilde{a}_i \wedge \tilde{x}_i\right)$ and $z = \tilde{\bigvee}_{j=1}^{m}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)$, from the fact that f_L is distributive, it follows that

$$\tilde{y} \tilde{\vee} \tilde{z} = \left(\left[\tilde{\bigvee}_{i=1}^{n}\left(\tilde{a}_i \wedge \tilde{x}_i\right)\right] \tilde{\vee} \left[\tilde{\bigvee}_{j=1}^{m}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)\right]\right) \in f_{L_0}.$$

$$\tilde{y} \tilde{\wedge} \tilde{z} = \left[\tilde{\bigvee}_{i=1}^{n}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right)\right] \tilde{\vee} \left[\tilde{\bigvee}_{j=1}^{m}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)\right]$$

Also, we get

$$= \tilde{\bigvee}_{i=1}^{n} \tilde{\bigvee}_{j=1}^{m} \left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right) \tilde{\wedge} \left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)$$

$$= \tilde{\bigvee}_{i=1}^{n} \tilde{\bigvee}_{j=1}^{m} \left(\tilde{a}_i \tilde{\wedge} \tilde{b}_j\right) \tilde{\wedge} \left(\tilde{x}_i \tilde{\wedge} \tilde{y}_j\right)$$

Noting that, $\tilde{a}_i \tilde{\wedge} \tilde{b}_j \in f_M$ and $\tilde{x}_i \tilde{\wedge} \tilde{y}_j \in \tilde{K}$, we obtain $\tilde{x} \tilde{\wedge} \tilde{y} \in f_{L_0}$. Depending on the same operations in f_L, it is seen that f_{L0} is a sublattice of f_L. f_{L0} contains f_M and $\tilde{K}$, because for any $\tilde{a}$ in f_M and $\tilde{x}$ in $\tilde{K}$, $\tilde{a} = \tilde{a} \tilde{\wedge} \bar{1}$ and $\tilde{x} = \bar{1} \tilde{\wedge} \tilde{x}$ lie in f_{L0}. $f_{L'}$ is a lattice in f_L, if $\tilde{K} \tilde{\subseteq} f_{L'}$ and $f_M \tilde{\subseteq} f_{L'}$ for $\tilde{\forall} \tilde{a}_i \in f_M$ and $\tilde{x}_i \in \tilde{K}, n \in N - \{0\}$, then

$$\tilde{a}_i \tilde{\wedge} \tilde{x}_i \in f_M \Rightarrow \tilde{\bigvee}_{i=1}^{n}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right) \in f_{L'} \Rightarrow f_{L_0} \tilde{\subseteq} f_{L'} \Rightarrow f_{L_0} = f_M\left(\tilde{K}\right).$$

Thus, f(K) is a minimal sublattice which contains $\tilde{K}$.

3.13 Lemma

Suppose that f_L is a soft fuzzy complete Heyting algebra such that $\max f_L = \bar{1}$ and $\min f_L = \bar{0}$. If f_M and $\tilde{K}$ are soft fuzzy complete Heyting sub algebras of f_L such that $\max f_M = \max \tilde{K} = \bar{1}$ and $\min f_M = \min \tilde{K} = \bar{0}$, then,

$$\overline{f_M\left(\tilde{K}\right)} = \left\{ \tilde{\bigvee}_{i=1}^{n} \left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right) : \tilde{a}_i \in f_M, \tilde{x}_i \in \tilde{K} \right\}$$

is the minimal soft fuzzy complete Heyting subalgebra of f_L that contains f_M and $\tilde{K}$ where $\bar{I}$ is an index set of arbitrary cardinality.

Proof: Let $\tilde{x}, \tilde{y} \in \overline{f_M\left(\tilde{K}\right)}$. Then,

$$\tilde{x} = \tilde{\bigvee}_{i \in I_1}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right), \tilde{y} = \tilde{\bigvee}_{i \in I_2}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right),$$

$$\tilde{x} \tilde{\vee} \tilde{y} = \left[\tilde{\bigvee}_{i \in I_1}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right)\right] \tilde{\vee} \left[\tilde{\bigvee}_{j \in I_2}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)\right],$$

$$= \tilde{\bigvee}_{i \in I_1 \cup I_2}\left(\tilde{c}_i \tilde{\wedge} \tilde{z}_i\right) \in \overline{f_M\left(\tilde{K}\right)}.$$

$$\tilde{x} \tilde{\wedge} \tilde{y} = \left[\tilde{\bigvee}_{i \in I_1}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right)\right] \tilde{\wedge} \left[\tilde{\bigvee}_{j \in I_2}\left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)\right].$$

Since f_L is a soft fuzzy complete Heyting algebra, then

$$\tilde{x} \tilde{\wedge} \tilde{y} = \tilde{\bigvee}_{\substack{i \in I_1 \\ j \in I_2}}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right) \tilde{\wedge} \left(\tilde{b}_j \tilde{\wedge} \tilde{y}_j\right)$$

$$= \tilde{\bigvee}_{(i,j) \in I_1 \times I_2}\left(\tilde{a}_i \tilde{\wedge} \tilde{b}_j\right) \tilde{\wedge} \left(\tilde{x}_i \tilde{\wedge} \tilde{y}_j\right)$$

Since $\tilde{K}$ is a soft fuzzy sublattice, then

$$\tilde{x} \tilde{\wedge} \tilde{y} \in \overline{f_M\left(\tilde{K}\right)}.$$

$\tilde{\bigvee}_i\left(\tilde{y}_i \tilde{\wedge} \tilde{z}\right) = \left(\tilde{\bigvee}_i \tilde{y}_i\right) \tilde{\wedge} \tilde{z}$. We also know that $\overline{f_M\left(\tilde{K}\right)} \tilde{\subseteq} f_L$ and that f_L is a soft fuzzy complete Heyting algebra, for $\tilde{y}_i \in \overline{f_M\left(\tilde{K}\right)}, \tilde{z}_i \in \overline{f_M\left(\tilde{K}\right)}$ *in* $\overline{f_M\left(\tilde{K}\right)}$. Thus

$$\tilde{\bigvee}_i\left(\tilde{y}_i \tilde{\wedge} \tilde{z}\right) = \left(\tilde{\bigvee}_i \tilde{y}_i\right) \tilde{\wedge} \tilde{z}$$

$$\tilde{A} = \left\{\tilde{y}_t \in \overline{f_M\left(\tilde{K}\right)} : t \in T\right\}$$

Where

$$\tilde{y}_t = \tilde{\bigvee}_{i \in I_t}\left(\tilde{a}_{t_i} \tilde{\wedge} \tilde{x}_{t_i}\right),$$

$$\tilde{\bigvee}_{t \in T} \tilde{y}_t = \tilde{\bigvee}_{t \in T}\left(\tilde{\bigvee}_{i \in I_t}\left(\tilde{a}_{t_i} \tilde{\wedge} \tilde{x}_{t_i}\right)\right)$$

Since f_L is a soft fuzzy complete Heyting algebra, then

$$\tilde{\bigvee}_{t \in T} \tilde{y}_t = \tilde{v}_{t \in T}\left(\tilde{v}_{t \in T}\left(\tilde{a}_{t_i} \tilde{\wedge} \tilde{x}_{t_i}\right)\right)$$

$$= \tilde{\bigvee}_{i \in \bigcup_{t \in T} I_t}\left(\tilde{a}_{t_i} \tilde{\wedge} \tilde{x}_{t_i}\right) \in \overline{f_M\left(\tilde{K}\right)}.$$

Then $\overline{f_M\left(\tilde{K}\right)}$ is called the soft fuzzy complete Heyting subalgebra.

If $\tilde{a} \in f_M$ *and* $\tilde{x} \in \tilde{K}$, then

$$\tilde{a} = (\tilde{a}\ EMBED\ Equation.\ DSMT4\bar{1}) \in \overline{f_M(\tilde{K})}\ and$$

$$\tilde{x} = (\tilde{1}\ EMBED\ Equation.\ DSMT4\bar{x}) \in \overline{f_M(\tilde{K})}.$$

Therefore,

$$f_M EMBED\ Equation.\ DSMT4)\overline{f_M(\tilde{K})}\ and$$

$$\tilde{K} = EMBED\ Equation.\ DSMT4\overline{f_M(\tilde{K})}.$$

Let $f_L{}^t$ be a complete soft fuzzy Heyting subalgebra and $f_M = EMBED\ Equation.\ DSMTf_{L^t})\ and$. Then $\tilde{x}_i \in \tilde{K}$ for $\tilde{K} = EMBED\ Equation.\ DSMT4 f_{L^t}$.

all $\tilde{a}_i \in f_M$. Thus, we get

$$\tilde{\vee}_{i\in I}\left(\tilde{a}_i \tilde{\wedge} \tilde{x}_i\right) \in f_{(L^{\tilde{t}})} \Rightarrow \overline{f_M\left(\tilde{K}\right)} \tilde{\subseteq} f_{(L^{\tilde{t}})}$$

This completes the proof that $f_L{}^t$, is the minimal complete soft fuzzy Heyting subalgebra of $\overline{f_M\left(\tilde{K}\right)}$ which contains f_M and $\tilde{K}$.

4. Results

In this study, some definitions and lemmas related to σ - algebra and measure in classical and fuzzy mathematics were defined on soft sets. Soft generalized fuzzy σ - algebra and soft generalized fuzzy measure were given using some definitions related to soft fuzzy De Morgan algebra and soft fuzzy sets. In addition, by using some examples and lemmas a theorem was proved.

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