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Observation on the Ternary Quadratic Diophantine Equation $x^2 = 2^{2k+1}(3z^2 - y^2)$

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Abstract

Objectives: Finding the integral solutions of ternary quadratic equation and the relation between parameters/solutions. The ternary quadratic Diophantine equation is studied for it's non-trivial distinct integral solutions. **Findings:** Employing the integral solutions of the equation under consideration infinitely many Pythagorean triangles are obtained where each of which satisfies the relation "Hypotenuse-4(area/Perimeter) is a Nasty number".

Keywords: Integral Solutions, Ternary Quadratic Equation

Notations

$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$	Polygonal number of rank n with sides m
$p_n^r = \frac{1}{6}n(n+1)(r-2)n + (5-r)$	Pyramidal number of rank n with sides r
$Pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$	Pentatope number of rank n
$SO_n = n(2n^2 - 1)$	Stella Octangula number of rank n
$\Pr_n = n(n+1)$	Pronic number of rank n
$gn_a = 2a - 1$	Gnomonic number of rank a
$OH_n = \frac{n}{3}(2n^2 + 1)$	Octahedral number of rank n
$Ct_{m,n} = (mn(n+1)+2)/2$	Centered Polygonal number of rank n with sidesm

1. Introduction

The problem of finding nonzero integral solutions for the quadratic homogeneous or non-homogeneous Diophantine equations have been an interest to many mathematicians since antiquity as can be seen from[1,2]. In [3], infinitely many Pythagorean triangles in each of which the hypotenuse is four times the product of the generator added with unity are obtained by employing the non-trivial integral solutions of the binary quadratic equation $y^2 = 3x^2 + 1$. In[4], the binary quadratic equation $y^2 = 10x^2 + 1$ is considered and by employing the nontrivial integral solutions of the equation, infinitely many Pythagorean triangles satisfying the relation$(2k^2 + 3k - 4)$ *Hypotenuse*–$(2k^2 + 2k - 5)$ one of the leg = (2k + 1) other leg + 1 are obtained. For other choices of Pythagorean triangles obtained through the integral solutions of corresponding ternary quadratic equations are given in[5–13]. Also[14–16] deals with the higher degree Diophantine equations and properties among their solutions. In this communication, we consider yet another ternary quadratic equation $x^2 = 2^{2k+1}(3z^2 - y^2)$ and illustrate a few properties among its solutions. Further making use of the integral solutions, patterns of Pythagorean triangles each with the property that "Hypotenuse - 4(Area/Perimeter) is a Nasty number" are obtained.

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2. Method of Analysis

The equation to be solved is

$$x^2 = 2^{2k+1}(3z^2 - y^2),\ k \in z^+$$

which is rewritten as

$$x^2 + 2^{2k+1}y^2 = 2^{2k}6z^2 \qquad (1)$$

Assuming z = z (k,a,b) $= a^2 + 2^{2k+1}b^2$ (2)

in(1), it is written in the factorizable form as

$$(x + i\sqrt{2^{2k+1}}\,y)(x - i\sqrt{2^{2k+1}}\,y) = (2^{k+1} + i\sqrt{2^{2k+1}}\,y)$$
$$(2^{k+1} - i\sqrt{2^{2k+1}}\,y)\ (a + i\sqrt{2^{2k+1}}\,b)^2 (a - i\sqrt{2^{2k+1}}\,b)^2$$

Define

$$(x + i\sqrt{2^{2k+1}}\,y) = (2^{k+1} + i\sqrt{2^{2k+1}}\,y)(a + i\sqrt{2^{2k+1}}\,b)^2$$

Equating real and imaginary parts the values of x and y are given by

$$\left.\begin{aligned} x(k,a,b) &= 2^{k+1}a^2 - 2^{3k+2}b^2 - 2ab2^{2k+1} \\ y(k,a,b) &= a^2 - 2^{2k+1}b^2 + 2^{k+2}ab \end{aligned}\right\} \qquad (3)$$

Thus (2) and (3), represent non-zero distinct integral solutions of (1). A few properties among the solutions are given below.

1. Each of the following expressions represents a Nasty number

   (i) $2^{k+1}y(k,2b,b) - x(k,2b,b)$

   (ii) $2^{k+1}[3*2^k z(k,a,b) - (2^k y(k,a,b) + x(k,a,b))]$

   (iii) 21 [2y(1,1,2) – x(1,1,2)]

2. $\dfrac{2^{k+1}y(k,b+1,b) - x(k,b+1,b)}{2^{2k}} = \Pr_b$

3. $2^k[z(2k,a,2a) + y(2k,a,2a) - x(2k,a,2a)] =$ sum of 2 perfect squares

4. $x(0,a,b) + y(0,a,b) + z(0,a,b) =$ difference of 2 perfect squares

6. $x(1,2b,b) + 4z(1,2b,b) = 0$

7. $xz(1,a,1) - t^2_{6,a} + 24P_a^5 - 11\Pr_a \equiv 1 \pmod 3$

3. Remarkable Observation

Taking y = n, and $x = 2^k(m-n)$, we get

$$m = 3a^2 - (2^{2k+2} + 2^{k+1})b^2$$

$$n = a^2 - 2^{2k+1}b^2 + 2^{k+2}ab$$

Considering m and n as the generators of the Pythagorean triangle, the sides of the triangle (p,q,r) are represented by

$$p(k,a,b) = 6a^4 + 2^{k+3}3a^3b - (2^{k+2}3 + 2^{2k+3} + 2^{k+2})a^2b^2$$
$$-(2^{3k+5} + 2^{2k+4})ab^3 + (2^{4k+4} + 2^{3k+3})b^4$$

$$q(k,a,b) = 8a^4 - 2^{k+3}a^3b + [-3(2^{2k+3} + 2^{k+2}) - 2^{2k+4}$$
$$+2^{2k+2}]a^2b^2 + 2^{3k+4}ab^3 + (2^{4k+4} + 2^{3k+4} + 2^{2k+2} - 2^{4k+2})b^4$$

$$q(k,a,b) = 8a^4 - 2^{k+3}a^3b + [-3(2^{2k+3} + 2^{k+2}) - 2^{2k+4}$$
$$+2^{2k+2}]a^2b^2 + 2^{3k+4}ab^3 + (2^{4k+4} + 2^{3k+4} + 2^{2k+2} - 2^{4k+2})b^4$$

in which $(a - 2^kb)^2 > (2^{2k+1} + 2^k)b^2$.

Here the Pythagorean triangle (p,q,r) satisfies the relation "Hypotenuse - (Area/Perimeter) is a Nasty number".As the expressions representing the sides of the Pythagorean triangle are cumbersome, we present below the properties satisfied by the sides of the Pythagorean triangle for the case k=0. For this case, the sides of the Pythagorean triangle are given by

$$\left.\begin{aligned} p(0,a,b) &= 6a^4 + 24a^3b - 24a^2b^2 - 48ab^3 + 24b^4 \\ q(0,a,b) &= 8a^4 - 8a^3b - 48a^2b^2 + 16ab^3 + 32b^4 \\ r(0,a,b) &= 10a^4 + 8a^3b - 24a^2b^2 - 16ab^3 + 40b^4 \end{aligned}\right\} \qquad (4)$$

where in $(b-a)^2 > 3b^2$

A few interesting properties satisfied by the above sides (4) are as follows:

1. Each of the following expressions represents a Nasty number:

(i) $p(0,1,r+2)-288\,\mathrm{P}_r^5-5730t_{3,r}+42$

(ii) $2[p(0,1,r+2)+r(0,1,r+2)-2q(0,1,r+2)$
$+456gn_r+192P_r^5+936]$

(iii) $33[p(0,1,r+2)+q(0,1,r+2)-r(0,1,r+2)$
$-384Pt_r-32P_r^5-40\,\mathrm{Pr}_r+44]$

2. $p(0,1,r+2)-24^2\,Pt_r\equiv 0(\mathrm{mod}\,6)$

3. $q(0,1,r+2)-240\,\mathrm{P}_r^4-72ct_{4,r}-32Pt_r\equiv 2(\mathrm{mod}\,3)$

4. $q(0,1,r+2)-24(gn_r*OH_r)-576\,\mathrm{P}_r^5-512\,\mathrm{Pr}_r\equiv 0(\mathrm{mod}\,4)$

5. $r(0,1,r+2)-10t_{6,r}*ct_{4,r}-728\,\mathrm{P}_r^5-872t_{3,r}\equiv 0(\mathrm{mod}\,2)$

6. $p(0,1,r+2)+r(0,1,r+2)-2q(0,1,r+2)+864t_{3,r}+192P_r^5$
$\equiv 0(\mathrm{mod}\,48)$

4. Conclusion

In conclusion one may search for other patterns of Pythagorean Triangles and their properties for different values of k.

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