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Discontinuous Galerkin Approximations for Volterra Integral Equations of the First Kind with Convolution Kernel

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Abstract

We presented the results of polynomial piece-wise approximations of discontinuous Galerkin (DG) for a Volterra integral equation of the first kind with kernel convolution where the kernel K is smooth and applies to the $K(0) \neq 0$ condition. We show that a DG approximation of m-th degree results in the total convergence of m degree if m is an odd number and when m is an even number, it gives a m + 1 degree. There is also a local hyper-convergence of one level higher order (For example, when m is odd, it is of m + 1 order and when m is even, it is of m + 1 order). But in the cases with even order, the hyper-convergence exists only when the exact solution u of the equation applies to $u^{(m+1)}(0) = 0$. We have also provided the numerical tests results that show that the theoretical convergence is optimal.

Keywords: Approximation Error, Convolution Kernel, Discontinuous Galerkin Approximations, Global Convergence, Local Superconvergence

1. Introduction

Given the increasing development of sciences in many fields, we observe the manifestation and application of integral equations in many engineering problems, physics, etc. For example, for the implementation of the single layer potential equations for the dispersion of sound of a surface, we are faced with Volterra integral equations of the first kind that for the development of its higher-order stepwise techniques, we need a simple approximation with simple analysis. Therefore in this context, we consider the discontinuous Galerkin approximation of piecewise polynomials for this type of equations with convolution kernel and evaluate its convergence[1].

Numerical tests also show that the approximations based on Discontinuous Galerkin (DG) has better stability than the one based on the conditions with the same locality. In this paper, first in Part 2 we derive discontinuous Galerkin approximation (DG) based on the matrix of the coefficients (We have discussed the properties of the matrices in Lemma 1). In Part 3, a fault equation is obtained for the approximations. In order to get an idea and use the easier method in Part 4, the convergence and hyper-convergence of the approximation for Volterra integral equation with special kernel of $K(t) \equiv 1$ was evaluated. Then, Part 5 analyzes the convergence of and hyper-convergence of approximations is general conditions. In Part 6, we presented the numerical tests results that show that the convergence is optimal and in Part 6, conclusions and discussions are provided[2–5].

2. Discontinuous Galerkin Approximation

In this research, we found new convergence results for piecewise polynomial discontinuous Galerkin approximations of a convolution kernel Volterra integral equation.

$$(\kappa u)(t) = \int_0^t K(t-s)u(s)ds = z(t) \quad t \in [0,T] \tag{1}$$

$$z \in C^{d+1}[0,T], K \in C^{d+1}[0,T], z(0) = 0, K(0) = 1 \tag{2}$$

that $K(0) \neq 0$.

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We want to find solutions u(t) of the convolution kernel. For, this matter we used d ≥ 0 to be specified. Brunner[1] considered that the equation (1) have a unique solution $u \in C^d[0, T]$. Also, we only satisfied approximations of equation (1) based on a $\Delta t = T/N$ for N > 0. Also, we can consider this equation:

$$S_m^{(-1)}(I_{\Delta t}) = \{v \in L^2(0,T) : v\big|_{I_n} \in P^{(m)}(I_n), n = 0 : N-1 \quad (3)$$

that $P^{(m)}(I_n)$ indicate the space of all real polynomials. It is appropriate to write again the approximate result on each subinterval as

$$U(t_m + q\Delta t) = \sum_{k=0}^{n} U_k^m P_k(2q-1) \quad for \quad q \in (0,1],$$

that k and the coefficients U_k^m for $k = 0:n$; $m = 0:N-1$ and P_k is the Legendre polynomial of degree. For each subinterval I_m we can defined m +1 basis functions by:

$$\varphi_k^m(t_m + q\Delta t) = \begin{cases} P_k(2q-1) & if \quad q \in (0,T] \\ 0 & otherwise \end{cases} \quad (3)$$

for $k = 0:n$. Hence,

$$U(t) = \sum_{m=0}^{N-1}\sum_{k=0}^{n} U_k^m \varphi_k^m(t) \quad for \quad t \in [0,T]. \quad (4)$$

We considered that the approximate result U(t) for $t \in [0, T]$ has been achieved, the DG approximation of equation (1) in the next subinterval I_m is found by substituting U given by equation (4) for u. We have:

$$< \varphi_j^m, z > = \sum_{\ell=0}^{N-1}\sum_{k=0}^{n} < \varphi_j^m, \kappa\varphi_k^\ell > U_k^\ell = \sum_{\ell=0}^{m}\sum_{k=0}^{n} < \varphi_j^m, \kappa\varphi_k^\ell > U_k^\ell$$

Also we can showed the complexity form of the equation (1) and the uniform mesh spacing imply that:

$$< \varphi_j^m, \kappa\varphi_k^\ell > = < \varphi_j^{m-\ell}, \kappa\varphi_k^0 > = \Delta t^2 \beta_{j,k}^{m-\ell}$$

That $\ell \le m$ and $j, k \ge 0$, so,

$$\beta_{j,k}^\ell = \begin{cases} \int_0^1 P_j(2s-1)\int_0^s K((s-s')\Delta t)P_k(2s'-1)ds'ds & when \quad \ell = 0 \\ \int_0^1 P_j(2s-1)\int_0^s K((\ell+s-s')\Delta t)P_k(2s'-1)ds'ds & when \quad \ell > 0 \end{cases}$$

If we consider:

$$z_j^m = \frac{1}{\Delta t^2} < \varphi_j^m, z > = \frac{1}{\Delta t}\int_0^1 P_j(2s-1)z(t_n + s\Delta t)ds$$

for $j \ge 0$ gives the complexity sum:

$$\sum_{\ell=0}^{m} B^{m-\ell}U^\ell = z^m \quad (5)$$

for U, where $(a^m)_j = a_j^m$ for $j = 0:m$ and $B^{m-\ell} \in \Re^{(m+1)\times(m+1)}$. For U^m in

$$B^0 U^m = z^n - \sum_{\ell=0}^{m-1} B^{m-\ell}U^\ell, \quad m = 0:N-1 \quad (6)$$

That B^0 is nonsingular the complete approximation.

3. An Equation for Discontinuous Galerkin Approximations Error

If we considered the approximate and exact results U(t) and u(t) convince the similar weak form of the VIE.

$$< \varphi_j^n, \kappa u > = < \varphi_j^n, a >, < \varphi_j^m, \kappa U > = < \varphi_j^m, a >, j = 0:m, m = 0:N-1,$$

Also, the approximation error, convinces the orthogonally equation

$$< \varphi_j^m, \kappa e > = 0 \quad for \quad j = 0:n, \quad m = 0:N-1.$$

So, we can be rewritten in the Legendre basis functions:

$$\int_0^1 P_j(2s-1)\Big\{\int_0^s K((s-s')\Delta t)e(t_m + s'\Delta t)ds' + \sum_{\ell=0}^{m-1}\int_0^1 K(t_{m-\ell} + (s-s')\Delta t) \times e(t_\ell + s'\Delta t)ds'\Big\}ds = 0 \quad (7)$$

that $j = 0:n$; $m = 0:N-1$. If $u \in C^{p+1}[0, T]$; $m \le N-1$, then

$$u(t_m + s\Delta t) = \hat{u}_{p,m}(s) + \tilde{R}_p^m(s),$$

Where

$$\hat{u}_{p,m}(s) = \sum_{k=0}^{p} \hat{u}_k^m P_k(2s-1)$$

$$\hat{u}_k^m = \frac{\int_0^1 P_k(2s-1)u(t_m + s\Delta t)ds}{\int_0^1 [P_k(2s-1)]^2 ds} \quad (8)$$

This result is the good L^2 degree p polynomial approximation of $u(t_m + s\Delta t)$, $s \in (0.1)$. The other term is achieved by

$$\left|\tilde{R}_p^m(s)\right| \le \frac{\left|u^{p+1}(\xi_m)\right|}{(p+1)!}\Delta t^{p+1} \qquad \xi_m(s) \in (t_n, t_{m+1}). \quad (9)$$

The $\hat{u}_k^m$ as a Legendre coefficient convince below relation:

$$\left|\hat{u}_k^m - \gamma_k u^k(t_{m+\frac{1}{2}})\Delta t^k\right| \le \frac{\Delta t^{M_{p,k}}(2s+1)}{2^{M_{p,k}} M_{p,k}!}\left\|u^{(M_{p,k})}\right\|_\infty, \quad (10)$$

that

$$\gamma = \frac{k!}{(2k)!} \quad for \quad k \ge 0; \quad M_{p,k} = \min\{k+2, p+1\} \quad (11)$$

Now we can show that:

$$\begin{aligned} e(t_m + s\Delta t) &= u(t_m + s\Delta t) - U(t_m + s\Delta t) \\ &= \sum_{k=0}^{p} \hat{u}_k^m P_k(2s-1) + \tilde{R}_p^m(s) - \sum_{k=0}^{n} U_k^m P_k(2s-1) \\ &= \sum_{k=0}^{n} \varepsilon_k^m P_k(2s-1) + \sum_{k=n+1}^{p} \hat{u}_k^m P_k(2s-1) + \tilde{R}_p^m(s) \quad (12) \end{aligned}$$

Where $\varepsilon_k^m = \hat{u}_k^m - U_k^m \quad for \quad k = 0:m, n \ge 0$

For the special case $P = n + 1$, we have:

$$\begin{aligned} e(t_m + s\Delta t) &= \sum_{k=0}^{n} \varepsilon_k^m P_k(2s-1) \\ &\quad + \hat{u}_{n+1}^n P_{n+1}(2s-1) + O(\Delta t^{n+2}) \\ &= \sum_{k=0}^{n} \varepsilon_k^m P_k(2s-1) + \gamma_{n+1} u^{n+1}(t_{m+\frac{1}{2}}) \\ &\quad \times \Delta t^{n+1} P_{n+1}(2s-1) + O(\Delta t^{n+2}) \quad (13) \end{aligned}$$

However, substituting equation (17) into equation (1) and rearranging gives:

$$\sum_{\ell=0}^{m} \mathrm{B}^{m-\ell}\varepsilon^\ell = b^m \quad for \quad m = 0:N-1 \quad (14)$$

the error vector is $\varepsilon^m = (\varepsilon_0^m, \varepsilon_1^m, \ldots, \varepsilon_m^m)^T$.

The right-hand side vector components:

$$\begin{aligned} b_j^m = -\int_0^1 P_j(2s-1)&\left\{\int_0^s K((s-s')\Delta t)\tilde{R}_p^m(s')ds' \right. \\ &\left. + \sum_{\ell=0}^{m-1}\int_0^1 K(t_{m-\ell} + (s-s')\Delta t)\tilde{R}_p^\ell(s')ds'ds\right\} \\ &- \sum_{k=n+1}^{p-1}\sum_{\ell=0}^{m} \beta_{j,k}^{m-\ell}\hat{u}_k^\ell \quad for \quad j = 0:m \end{aligned}$$

Also, If equation (2) preserve with d = p + 1, that $p \ge n + 1$, then

$$b_j^m = -\sum_{k=n+1}^{p-1}\gamma_k \Delta t^k\left(O(\Delta t^2) + \sum_{\ell=0}^{m}\beta_{j,k}^{m-\ell}u^k(t_{\ell+\frac{1}{2}})\right) + O(\Delta t^p) = O(\Delta t^{n+1}),$$

4. Convergence and Super Convergence for Special Case: K(t) ≡ 1

Here, we study the optimal convergence order of the DG method for Volterra integral equations with constant Kernel to get the main idea of convergence analysis.

When K(t) ≡ 1, the coefficient matrices in the error equation (18) simplify greatly and are given by (10) and, $B^0 = A^1$ for all $\ell \ge 1$ and so (18) becomes

$$A^0\varepsilon^m + A^1\sum_{\ell=0}^{m-1}\varepsilon^\ell = b^m \quad for \quad m = 0:N-1$$

$$A^0\varepsilon^{m+1} + (A^1 - A^0)\varepsilon^m = b^{m+1} - b^m \quad m = 0:N-2$$

that can be rewritten as

$$\varepsilon^{m+1} = G\varepsilon^m + (A^0)^{-1}(b^{m+1} - b^m), \quad (15)$$

In next step, we have

$$\varepsilon^{m+2} = G^2\varepsilon^m + g^m \quad for \quad m = 0:N-3 \quad (16)$$

$$g^m = (A^0)^{-1}(b^{m+2} - b^{m+1}) + G(A^0)^{-1}(b^{m+1} - b^m). \quad (17)$$

When k(t) ≡ 1 by bounding the approximation error e(t) = u(t) − U(t), the basic result of this subsection is to show that the Discontinuous Galerkin scheme for equation (1) converges.

5. Result

In this research, we investigate the test of conditions that we presented and various results obtained by convergence and hyper-convergence for $t \in [0,1]$.

$$\int_0^t J_0(\omega(t-s))u(s)ds = z(t) \quad for \quad t \in [0,1] \quad (18)$$

where J_0 is the zeroth-order Bessel function of the first kind, the oscillation frequency parameter $\omega > 0$, the $z(0) = 0$. The solution of equation (18) is

$$u(t) = z'(t) + \omega\int_0^t \frac{J_1(\omega s)}{s} z(t-s)ds,$$

J_1 is the first-order Bessel function. We can show for very high accuracy by numerical quadrature in the especial cases shown in Table 1. Also, all the required derivatives of k and z exist in the Table 1, $u^{m+1}(0)$ can be showed from the linear equations (3) with $p = 1 : m + 2$.

We show results in Figure 1 for data of Table 1. Comparing with the L_∞ error in the approximate result with the maximum error at the super convergence points.

Table 1. Especial cases and characteristics of the exact solution of at t = 0.

No	z(t)	ω	α	$u^{(1)}(0)$	$u^{(3)}(0)$	$u^{(5)}(0)$
1	$te^{-\alpha t^2}/(1+t^2)$	$\sqrt{24\alpha-24}$	$1+\sqrt{5}$	0	0	0
2	$te^{-\alpha t^2}/(1+t)$	$\sqrt{24\alpha-24}$	$1+\sqrt{5}$	−2	0	0
3	$t^3e^{-\alpha t^2}/(1+t)$	$\sqrt{120}$	3	0	−24	0
4	$te^{-\alpha t^2}/(1+t)$	$\sqrt{24\alpha}$	$-3+\sqrt{19}$	−2	−24	0
5	$t^5e^{-\alpha t^2}/(1+t)$	$\sqrt{120}$	3	0	0	−720

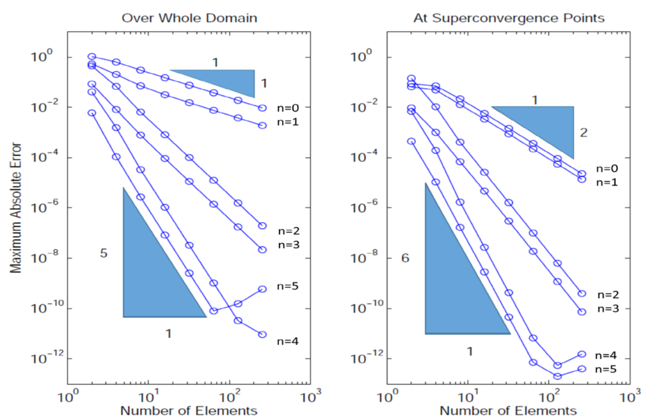

Figure 1. Solutions for data of Table 1 for approximations of degree $n = 0{:}5$.

6. Conclusion

If K and Z are appropriately flat, then the n^{th} degree Discontinuous Galerkin method for equation (1) converges to the exact solution. If n is odd, $O(\Delta t^n)$ as the order of convergence, and for each subinterval, $O(\Delta t^{n+1})$ as the local super convergence at the zeros of P'_{n+1}. If n is even, $O(\Delta t^{n+1})$ is convergence of order. Also, if the exact solution satisfies $u^{n+1}(0) = 0$, $O(\Delta t^{m+2})$ as an order of local super convergence at the zeros of P'_{n+2}. The numerical results that we found in this study certify the divination of the theory and also determined that convergence rates are optimal.

7. References

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