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Some Results on Fuzzy Hyper BCK-ideals of Hyper CK-algebras

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Abstract

The fuzzification of (weak, strong, reflexive) hyper BCK-positive implicative ideals in hyper BCK-algebras is considered. It is shown that every fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal is a fuzzy (weak, strong, reflexive) hyper BCK-ideal. Relations among weak hyper BCK-positive implicative ideals, hyper BCK-positive implicative ideals, strong hyper BCK-positive implicative ideals and reflexive hyper BCK-positive implicative ideals are given. It is proved that the product of fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideals is also a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal. Lastly the relations among fuzzy (weak, strong, reflexive) hyper BCK-(implicative, positive implicative, commutative) ideals are being discussed.

Keywords: Hyper Bck-Algebra, (Fuzzy) Hyper Bck-Positive Implicative Ideal, (Fuzzy) Reflexive Hyper Bck-Positive Implicative Ideal, (Fuzzy) Strong Hyper Bck-Positive Implicative Ideal, (Fuzzy) Weak Hyper Bck-Positive Implicative Ideal

1. Introduction

Imai and Iseki[4] in 1966, introduced the notion of BCK-algebras. After the introduction of the concept of fuzzy sets by Zadeh[11], various researchers discussed the idea of fuzzification of ideals in BCK-algebras and their generalizations. Dudek et al[3] discussed the properties of fuzzy BCC-ideals in BCC-algebras and their images.

Hyperstructure theory was introduced by Marty[10] in 1934 at the 8th congress of Scandinavian Mathematicians. This theory has wide applications in many different fields. Jun et al[8] in 2000, applied the hyperstructure theory to BCK-algebras and introduced the concept of hyper BCK-algebras, which is a generalization of BCK-algebras. Jun and Shim[5] applied the fuzzy set theory to the strong implicative hyper BCK-ideals in hyper BCK-algebras and discussed the relations among fuzzy strong hyper BCK-ideals, fuzzy implicative hyper BCK-ideals and fuzzy strong implicative hyper BCK-ideals. Borzooei et al[1] introduced the concept of hyper BCC-algebras, as a generalization of BCC-algebras. In this paper, we introduce the concept of fuzzification of (weak, strong, reflexive) hyper BCK-positive implicative ideals in hyper BCK-algebras and discuss some of their properties. The relations among fuzzy (weak, strong, reflexive) hyper BCK-(implicative, positive implicative, commutative) ideals will also be discussed.

2. Preliminaries

Let H be a non-empty set endowed with a hyper operation "o", that is, o is a function from $H\times H$ to PH, where PH is the collection of all non-empty subsets of H. For two subset A and B of H, denote by AoB the set $\{aob \mid a\in A, b\in B\}$. We shall use xoy instead of $xo\{y\}$, $\{x\}oy$ or $\{x\}o\{y\}$.

Definition 2.1.

By a hyper BCK-algebra we mean a non-empty set H endowed with a hyperoperation "o" and a constant 0 satisfying the following axioms[8]:

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$(HK1)$ $(xoz)o(yoz) \ll xoy$,
$(HK2)$ $(xoy)oz=(xoz)oy$,
$(HK3)$ $xoH \ll \{x\}$,
$(HK4)$ $x \ll y$ and $y \ll x$ imply $x=y$,
for all $x,y,z \in H$, where $x \ll y$ is defined by $0 \in xoy$ and for every A, $B \subseteq H$, $A \ll B$ is defined by $\forall\ a \in A$, $\exists\ b \in B$ such that $a \ll b$. In such case we call "$\ll$" the hyper order in H.

Proposition 2.2.

In any hyper BCK-algebra H, the following hold[8]:

(i) $xo0=\{x\}$, (vii) $Ao\{0\}=\{0\}$ implies $A=\{0\}$,
(ii) $xoy \ll x$, (viii) $0 \ll x$,
(iii) $0oA=\{0\}$, (ix) $0ox=\{0\}$,
(iv) $A \ll A$, (x) $0o0=\{0\}$,
(v) $A \subseteq B$ implies $A \ll B$, (xi) $y \ll z$ implies $xoz \ll xoy$,
(vi) $AoB \ll A$,

For all $x,y,z \in H$ and for all non-empty subsets A and B of H.

Let I be a non-empty subset of hyper BCK-algebra H and $0 \in I$. Then I is called a *hyper BCK-subalgebra* of H if $xoy \subseteq I$, for all $x,y \in I$, a *weak hyper BCK-ideal* of H if $xoy \subseteq I$ and $y \in I$ imply $x \in I$, for all $x,y \in H$, a *hyper BCK-ideal* of H if $xoy \ll I$ and $y \in I$ imply $x \in I$, for all $x,y \in H$, a *strong hyper BCK-ideal* of H if $xoy \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$, for all $x,y \in H$. I is said to be *reflexive* if $xox \subseteq I$ for all $x \in H$.

Lemma 2.3.

In a hyper BCK-algebra[6, 8]

- reflexive hyper BCK-ideal is a strong hyper BCK-ideal,
- strong hyper BCK-ideal is a hyper BCK-ideal,
- hyper BCK-ideal is a weak hyper BCK-ideal.

Lemma 2.4.

Let I be a reflexive hyper BCK-ideal of a hyper BCK-algebra H[6]. Then $xoy \cap I \neq \emptyset$ implies $xoy \ll I$, $\forall x$, $y \in H$.

Proposition 2.5.

Let A be a subset of a hyper BCK-algebra H[6]. If I is a hyper BCK-ideal of H such that $A \ll I$ then $A \subseteq I$.

Definition 2.6.

Let H be a hyper BCK-algebra. A non-empty subset $I \subseteq H$ containing 0 is called

- a weak hyper BCK-positive implicative ideal of H if for all $x,y,z \in H$,
  $((xoz)oz)o(yoz) \subseteq I$ and $y \in I$ imply $(xoz) \subseteq I$.
- a hyper BCK-positive implicative ideal of H if for all $x,y,z \in H$,
  $((xoz)oz)o(yoz) \ll I$ and $y \in I$ imply $(xoz) \subseteq I$.
- a strong hyper BCK-positive implicative ideal of H if for all $x,y,z \in H$,
  $((xoz)oz)o(yoz) \cap I \neq \emptyset$ and $y \in I$ imply $(xoz) \subseteq I$.

It is not difficult to see that the following proposition is true.

Propostion 2.7.

Every (weak, strong, reflexive) hyper BCK-positive implicative ideal of a hyper BCK-algebra H is a (weak, strong, reflexive) hyper BCK-ideal of H.

Theorem 2.8.

In any hyper BCK-algebra

(i) Hyper BCK-positive implicative ideal is a weak hyper BCK-positive implicative ideal.
(ii) Strong hyper BCK-positive implicative ideal is a hyper BCK-positive implicative ideal.
(iii) Reflexive hyper BCK-positive implicative ideal is a strong hyper BCK-positive implicative ideal.

Proof. (i) Suppose that I is a hyper BCK-positive implicative ideal of a hyper BCK-algebra H.

For any $x,y,z \in H$, let $((xoz)oz)o(yoz) \subseteq I$ and $y \in I$. Then $((xoz)oz)o(yoz) \subseteq I$ implies $((xoz)oz)o(yoz) \ll I$ (by Proposition 2.2(v)), which along with $y \in I$ implies $xoz \subseteq I$. Hence I is a weak hyper BCK-positive implicative ideal of H.

(ii) Suppose that I is a strong hyper BCK-positive implicative ideal of a hyper BCK-algebra H. Let $((xoz)oz)o(yoz) \ll I$ and $y \in I$. Then for all $a \in ((xoz)oz)o(yoz), \exists b \in I$ such that $a \ll b$. This implies $0 \in aob$ and thus $aob \cap I \neq \emptyset$. By Proposition 2.7, I is a strong hyper BCK-ideal of H. Therefore $aob \cap I \neq \emptyset$ along with $b \in I$ implies $a \in I$, that is $((xoz)oz)o(yoz) \subseteq I$. Therefore $(((xoz)oz)o(yoz)) \cap I \neq \emptyset$, which along with $y \in I$ implies $xoz \subseteq I$. Hence I is a hyper BCK-positive implicative ideal of H.

(iii) Suppose that I is a reflexive hyper BCK-positive implicative ideal of a hyper BCK-algebra H. For any $x,y,z \in H$, let $(((xoz)oz)o(yoz)) \cap I \neq \emptyset$ and $y \in I$. Being a reflexive hyper BCK-positive implicative ideal, I is also a reflexive hyper BCK-ideal of H (by Proposition 2.7), therefore by Lemma 2.4, from $(((xoz)oz)o(yoz)) \cap I \neq \emptyset$, we obtain, $((xoz)oz)o(yoz) \ll I$, which along with $y \in I$ implies $xoz \subseteq I$. Hence I is a strong hyper BCK-positive implicative ideal of H.

Definition 2.9.

Let H be a hyper BCK-algebra[7]. A fuzzy subset μ, i.e., the map $\mu{:}H\rightarrow[0,1]$, is called

- a fuzzy weak hyper BCK-ideal of H if for all $x,y\in H$,
  $\mu(0)\geq\mu(x)\geq\min\{\inf_{a\in xoy}\mu(a),\mu(y)\}$
- a fuzzy hyper BCK-ideal of H if $x\ll y$ implies $\mu(x)\geq\mu(y)$ and for all $x,y\in H$,
  $\mu(x)\geq\min\{\inf_{a\in xoy}\mu(a),\mu(y)\}$
- a fuzzy strong hyper BCK-ideal of H if for all $x,y\in H$,
  $\inf_{a\in xox}\mu(a)\geq\mu(x)\geq\min\{\sup_{b\in xoy}\mu(b),\mu(y)\}$
- a fuzzy reflexive hyper BCK-ideal of H if for all $x,y\in H$,
  $\inf_{a\in xox}\mu(a)\geq\mu(y)$ and $\mu(x)\geq\min\{\sup_{b\in xoy}\mu(b),\mu(y)\}$

It is not difficult to see that the following theorem is true.

Theorem 2.10.

In any hyper BCK-algebra[7]

- Fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal.
- Fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal.
- Fuzzy reflexive hyper BCK-ideal is a fuzzy strong hyper BCK-ideal.

Definition 2.11.

Let H be a hyper BCK-algebra and let $t\in xo(yo(yox))$. A fuzzy set μ in H is called

- a fuzzy weak hyper BCK-implicative (resp. BCK-commutative) ideal of H if $\mu(0)\geq\mu(x)$ and $\mu(t)\geq\min\{\inf_{a\in((xoy)oy)oz}\mu(a),\mu(z)\}$, (resp. $\mu(t)\geq\min\{\inf_{a\in((xoy)oz}\mu(a),\mu(z)\}$) for all $x,y,z\in H$.
- a fuzzy hyper BCK-implicative (resp. BCK-commutative) ideal of H if $x<<y$ implies $\mu(x)\geq\mu(y)$ and $\mu(t)\geq\min\{\inf_{a\in((xoy)oy)oz}\mu(a),\mu(z)\}$, (resp. $\mu(t)\geq\min\{\inf_{a\in((xoy)oz}\mu(a),\mu(z)\}$) for all $x,y,z\in H$.
- a fuzzy strong hyper BCK-implicative (resp. BCK-commutative) ideal of H if $\inf_{a\in xox}\mu(a)\geq\mu(x)\}$ and $\mu(t)\geq\min\{\sup_{b\in((xoy)oy)oz}\mu(b),\mu(z)\}$, (resp. $\mu(t)\geq\min\{\sup_{b\in(xoy)oz}\mu(b),\mu(z)\}$) for all $x,y,z\in H$.
- a fuzzy reflexive hyper BCK-implicative (resp. BCK-commutative) ideal of H if $\inf_{a\in xox}\mu(a)\geq\mu(y)$ and $\mu(t)\geq\min\{\sup_{b\in((xoy)oy)oz}\mu(b),\mu(z)\}$, (resp. $\mu(t)\geq\min\{\sup_{b\in(xoy)oz}\mu(b),\mu(z)\}$) for all x,y,z$\in$H.

3. Fuzzy Hyper BCK-positive Implicative Ideals

Now we introduce the notions of fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideals in hyper BCK-algebras and discuss some of their properties.

Definition 3.1.

Let H be a hyper BCK-algebra and let $t\in xoz$. A fuzzy set μ in H is called

- a fuzzy weak hyper BCK-positive implicative ideal of H if $\mu(0)\geq\mu(x)$ and $\mu(t)\geq\min\{\inf_{a\in((xoz)oz)o(yoz)}\mu(a),\mu(y)\}$ for all $x,y,z\in H$.
- a fuzzy hyper BCK-positive implicative ideal of H if $x<<y$ implies $\mu(x)\geq\mu(y)$ and $\mu(t)\geq\min\{\inf_{a\in((xoz)oz)o(yoz)}\mu(a),\mu(y)\}$ for all $x,y,z\in H$.
- a fuzzy strong hyper BCK-positive implicative ideal of H if $\inf_{a\in xox}\mu(a)\geq\mu(x)$ and $\mu(t)\geq\min\{\sup_{b\in((xoz)oz)o(yoz)}\mu(b),\mu(y)\}$ for all $x,y,z\in H$.
- a fuzzy reflexive hyper BCK-positive implicative ideal of H if $\inf_{a\in xox}\mu(a)\geq\mu(y)$ and $\mu(t)\geq\min\{\sup_{b\in((xoz)oz)o(yoz)}\mu(b),\mu(y)\}$ for all x,y,z$\in$H.

Theorem 3.2.

Let H be a hyper BCK-algebra. Then every fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal of H is a fuzzy (weak, strong, reflexive) hyper BCK-ideal of H.

Proof. Let μ be a fuzzy hyper BCK-positive implicative ideal of H. Then for any $x,y,z\in H$ and for all $t\in xoz$ we have,

$$\mu(t)\geq\min\{\inf_{a\in((xoz)oz)o(yoz)}\mu(a),\mu(y)\}$$

Putting $z=0$ we get,

$$\mu(t)\geq\min\{\inf_{a\in((xo0)o0)o(yo0)}\mu(a),\mu(y)\}$$

which gives,

$$\mu(x)\geq\min\{\inf_{a\in xoy}\mu(a),\mu(y)\}$$

Thus μ is a fuzzy hyper BCK-ideal of H.

Let μ be a fuzzy set in a hyper BCK-algebra H. Then the set defined by $\mu_t=\{x\in H{:}\mu(x)\geq t\}$, where $t\in[0,1]$, is called a level subset of H.

The transfer principle for fuzzy sets described in[9] suggest the following theorem.

Theorem 3.3.

Let μ be a fuzzy set in a hyper BCK-algebra H. Then μ is a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal of H if and only if for all $t\in[0,1]$, $\mu t\neq\emptyset$ is a (weak, strong, reflexive) hyper BCK-positive implicative ideal of H.

Proof. Suppose that m is a fuzzy hyper BCK-positive implicative ideal of H. Since $\mu t \neq \varnothing$, so for any $x \in \mu_t$, $\mu(x) \geq t$. Since a fuzzy hyper BCK-positive implicative ideal is a fuzzy hyper BCK-ideal (by Theorem 3.2) and a fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal (by Theorem 2.10), so μ is also a fuzzy weak hyper BCK-ideal of H. Thus $\mu(0) \geq \mu(x) \geq t$, for all $x \in H$, which implies $0 \in \mu_t$.

Let $((xoz)oz)o(yoz) << \mu_t$ and $y \in \mu_t$, for some $x,y,z \in H$. Then for all $a \in ((xoz)oz)o(yoz), \exists b \in \mu_t$ such that $a << b$. So $\mu(a) \geq \mu(b) \geq t$, for all $a \in ((xoz)oz)o(yoz)$. Thus $\inf_{a \in ((xoz)oz)o(yoz)} \mu(a) \geq t$. Also $\mu(y) \geq t$, as $y \in \mu_t$. Therefore for all $v \in xoz$, μ satisfies

$$\Rightarrow v \in \mu_t \Rightarrow xoz \subseteq \mu_t$$

Hence μ_t is hyper BCK-positive implicative ideal of H.

Conversely suppose that $\mu_t \neq \varnothing$ is a hyper BCK-positive implicative ideal of H for all $t \in [0,1]$. Let $x << y$ for some $x,y \in H$ and put $\mu(y)=t$. Then $y \in \mu_t$. So $x << y \in \mu t \Rightarrow x << \mu_t$. Being a hyper BCK-positive implicative ideal, μ_t is also a hyper BCK-ideal of H (by Proposition (2.7)) therefore by Proposition 2.5, $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. That is $x << y \Rightarrow \mu(x) \geq \mu(y)$, for all $x,y \in H$.

Moreover for any $x,y,z \in H$, let $d = min\ \{inf_{c \in ((xoz)oz)o(yoz)} \mu(c), \mu(y)\}$. Then $\mu(y) \geq d \Rightarrow y \in \mu_d$ and for all $e \in ((xoz)oz)o(yoz)$, $\mu(e) inf_{c \in ((xoz)oz)o(yoz)} \mu(c) \geq d$, which implies that $e \in \mu_d$.

Thus $((xoz)oz)o(yoz) \subseteq \mu_d$.

By Proposition 2.2(v), $((xoz)oz)o(yoz) \subseteq \mu_d \Rightarrow ((xoz)oz)o(yoz) << \mu_d$, which along with $y \in \mu_d$ implies $xoz \subseteq \mu_d$. Hence for all $u \in xoz$, we get

$$\mu(u) \geq d = min\{inf_{c \in ((xoz)oz)o(yoz)} \mu(c), \mu(y)\}$$

Thus μ is a fuzzy hyper BCK-positive implicative ideal of H.

The following corollary is a simple consequence of Theorem 3.3.

Corollary 3.4.

For any subset A of a hyper BCK-algebra H, let μ be a fuzzy set in H defined by:

$$\mu(x) = \begin{cases} t \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

for all $x \in H$, where $t \in (0,1]$. Then A is a (weak, strong, reflexive) hyper BCK-positive implicative ideal of H if and only if μ is a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal of H.

4. Product of Fuzzy Hyper BCK-positive Implicative Ideals

Definition 4.1.

Let $(H_1, o_1, 0_1)$ and $(H_2, o_2, 0_2)$ are hyper BCK-algebras[2] and $H = H_1 \times H_2$. We define a hyperoperation "o" on H by:

$$(a_1,b_1)o(a_2,b_2) = (a_1 o a_2,\ b_1 o b_2)$$

for all $(a_1,b_1),(a_2,b_2) = H$, where for $A \subseteq H$ and $B \subseteq H$ by $A \times B$ we mean,

$$A \times B = \{(a,b) : a \in A, b \in B\}.$$

and $0 = (0_1, 0_2)$ and a hyperorder "<<" on H by:

$$(a_1,b_1) << (a_2,b_2) \Leftrightarrow a_1 << a_2 \text{ and } b_1 << b_2.$$

Thus $(H,o,0)$ is a hyper BCK-algebra.

Let μ and v be fuzzy sets in hyper BCK-algebras H_1 and H_2 respectively. Then $\mu \times v$, the product of μ and v of $H = H_1 \times H_2$ is defined as:

$$(\mu \times \nu)((x,y)) = \min\ \{\mu(x), \nu(y)\}.$$

From now on, let H_1 and H_2 are hyper BCK-algebras and let $H = H_1 \times H_2$.

Definition 4.2.

Let μ be a fuzzy set in H. Then fuzzy sets μ_1 and μ_2 on H_1 and H_2 respectively, are defined as:

$$\mu_1(x) = \mu((x,0)),\ \mu_2(y) = \mu((0,y)).$$

Theorem 4.3.

Let μ be a fuzzy set in H. If μ is a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal of H, then $\mu = \mu_1 \times \mu_2$, where μ_1 and μ_2 are fuzzy sets on H_1 and H_2 respectively.

Proof. Suppose that μ is a fuzzy hyper BCK-positive implicative ideal of H. Then for any (x,u), (y,v), $(z,w) \in H$, where $x,y,z \in H_1$ and $u,v,w \in H_2$ and for all $(a,b) \in (x,u)$ $o(z,w) = (xoz,\ uow)$, we have

Putting $d=y=z=w=0$ and $v=u$, we get

.

Thus,

,

this implies, $\mu((x,u)) \geq \min\ \{\mu_1(x), \mu_2(u)\}$

that is, $\mu((x,u)) \geq (\mu_1 \times \mu_2)((x,u))$

Therefore we get, $\mu_1 \times \mu_2 \subseteq \mu$ (1)

Conversely, since $(x,0) << (x,u)$ and $(0,u) << (x,u)$.

This gives, $\mu((x,0))\geq\mu((x,u))$ and $\mu((0,u))\geq\mu((x,u))$

Therefore,

$\min\{\mu(x,u), \mu(x,u)\}= \mu(x,u)$.

Which implies, $(\mu_1\times\mu_2)((x,u))\geq\mu(x,u)$,

that is, $\mu\subseteq\mu_1\times\mu_2$ (2)

Hence from (1) and (2) we have, $\mu_1\times\mu_2=\mu$

Theorem 4.4.

Let m be a fuzzy set in H. Then μ is a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal of H if and only if and are fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideals of and respectively.

Proof. Let μ be a fuzzy hyper BCK-positive implicative ideal of H and let $x_1<<x_2$ for some $x_1x_2\in H_1$. Then $(x_1,0)<<(x_2,0)$ which implies $\mu((x_1,0))=\mu_1(x_1)\geq\mu((x_2,0))=\mu_1(x_2)$, that is, $\mu_1(x_1)\geq\mu_1(x_2)$

Moreover for any $x_1y_1z_1\in H_1$,

Let $t=\min\{\inf_{a\in((x_1oz_1)oz_1)o(y_1oz_1)} \mu_1(a), \mu_1(y_1)\}$

Then for all $b\in((x_1oz_1)oz_1)o(y_1oz_1)$,

$\mu_1(b)\geq\inf_{a\in((x_1oz_1)oz_1)o(y_1oz_1)} \mu_1(a)\geq t$ and $\mu_1(y_1)\geq t$.

This implies, $\mu((b,0))\geq t$ and $\mu((y_1,0))\geq t$,

for all $(b,0)\in(((x_1,0)o(z_1,0))o(z_1,0))o(y_1,0)o(z_1,0))$

Therefore, $(b,0)\in\mu_t$ and $(y_1,0)\in\mu_t$,

that is, $(((x_1,0)o(z_1,0))o(z_1,0))o((y_1,0)o(z_1,0))\subseteq\mu_t$ and $(y_1,0)\in\mu_t$,

Since by Theorem 3.3, $\mu_t\neq\emptyset$ is a hyper BCK-positive implicative ideal of H and so is a weak hyper BCK-positive implicative ideal of H (by Theorem 2.8(i)). Thus $(((x_1,0)o(z_1,0))o(z_1,0))o((y_1,0)o(z_1,0))\subseteq\mu_t$ and $(y_1,0)\in\mu_t$ imply $(x_1,0)o(z_1,0)\subseteq\mu_t$

Therefore $\mu((s,0))\geq t$, for all $(s,0)\in(x_1,0)o(z_1,0)=(x_1oz_1,0)$, that is, $\mu_1(s)\geq t=\min\{\inf_{a\in((x_1oz_1)oz_1)(y_1oz_1)} \mu_1(a),\mu_1(y_1)\}$, for all

Hence μ_1 is a fuzzy hyper BCK-positive implicative ideal of H_1.

Similarly we can prove that is a fuzzy hyper BCK-positive implicative ideal of H_2.

Conversely suppose that μ_1 and μ_2 are fuzzy hyper BCK-positive implicative ideals of H_1 and H_2 respectively.

For any $(x,u),(y,v)\in H$, where $x,y\in H_1$ and $u,v\in H_2$, let $(x,u)<<(y,v)$

Since $(x,u)<<(y,v)\Leftrightarrow x<<y$ and $u<<v$

Thus $\mu_1(x)\geq\mu_1(y)$ and $\mu_2(u)\geq\mu_2(v)$

So, $\min\{\mu_1(x), \mu_2(u)\}\geq \min\{\mu_1(y), \mu_2(v)\}$

this implies, $(\mu_1\times\mu_2)((x,u))\geq(\mu_1\times\mu_2)((y,v))$

that is, $\mu((x,u))\geq\mu((y,v))$ (by Theorem 4.3, $\mu=\mu_1\times\mu_2$)

Therefore, $(x,u)<<(y,v)$ implies $\mu((x,u))\geq\mu((y,v))$

Moreover for any (x,u), (y,v), $(z,w)\in H$, where $x,y,z\in H_1$ and $u,v,w\in H_2$ and for all $(a,b)\in(x,u)o(z,w)=(xoz, uow)$, we have

$\mu((a,b))=(\mu_1\times\mu_2)((a,b))=\min\{\mu_1(a), \mu_2(b)\}$

this implies,

Hence μ is a fuzzy hyper BCK-positive implicative ideal of H. **Relationship between fuzzy (weak, strong, reflexive) hyper BCK-(implicative, positive implicative, commutative) ideals**

Now we discuss the relationship among fuzzy (weak, strong, reflexive) hyper BCK-(implicative, positive implicative, commutative) ideals of a hyper BCK-algebra and show that a fuzzy set μ in a hyper BCK-algebra H is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ is both a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal and a fuzzy (weak, strong, reflexive) hyper BCK-commutative ideal of H.

Theorem 4.5.

A fuzzy set μ in a hyper BCK-algebra H is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ is both a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal and a fuzzy (weak, strong, reflexive) hyper BCK-commutative ideal of H.

Proof. Let μ be a fuzzy hyper BCK-implicative ideal of a hyper BCK-algebra H. Then for any $x,y,z\in H$ and for all $t\in xo(yo(yox))$,

$\mu(t)\geq\min\{\inf_{a\in((xoy)oy)oz} \mu(a), \mu(z)\}$.

Since $((xoy)oy)oz=((xoy)oz)oy\ll(xoy)oz$ (by ($HK2$) and Proposition 2.2(vi))

Then for all $a\in((xoy)oy)oz$, $\exists$ $b\in(xoy)oz$ such that $a\ll b$.

This implies, $\mu(a)\geq\mu(b)$ for all $a\in((xoy)oy)oz$ and for some $b\in(xoy)oz$,

that is, $\inf_{a\in((xoy)oy)oz}\mu(a)\geq\mu(b)\geq\inf_{c\in(xoy)oz}\mu(c)$.

Thus for all $t\in xo(yo(yox))$,

.

Hence μ is a fuzzy hyper BCK-commutative ideal of H.

Moreover, $yo(yox)\ll y$ implies $xoy\ll xo(yo(yox))$ (by Proposition 2.2 (vi, xi))

Then for all $s\in xoy$, $\exists$ $t\in xo(yo(yox))$ such that $s\ll t$.

Thus $\mu(s)\geq\mu(t)$ for all $s\in xoy$ and for some $t\in xo(yo(yox))$.

Also, $zoy\ll z$ implies, $((xoy)oy)oz\ll((xoy)oy)o(zoy)$.

Then for all $a\in((xoy)oy)oz$, $\exists$ $b\in((xoy)oy)o(zoy)$ such that $a\ll b$.

Therefore $\mu(a)\geq\mu(b)$ for all $a\in((xoy)oy)oz$ and some $b\in((xoy)oy)o(zoy)$,

that is, $\inf_{a\in((xoy)oy)oz}\mu(a)\geq\mu(b)\geq\inf_{c\in((xoy)oy)o(zoy)}\mu(c)$.

Thus for all $s\in xoy$,

,

which gives, $\mu(s) \geq \min\{\inf_{c\in((xoy)oy)o(zoy)}\mu(c), \mu(z)\}$.
Hence μ is a fuzzy hyper BCK-positive implicative ideal of H.
Conversely suppose that μ is both a fuzzy hyper BCK-commutative ideal and a fuzzy hyper BCK-positive implicative ideal of H.
Then for any $x,y,z\in H$ and for all $t\in xo(yo(yox))$,

$\mu(t) \geq \min\{\inf_{c\in(xoy)oz}\mu(a), \mu(z)\}$.

Since μ is also a fuzzy hyper BCK-positive implicative ideal of H, thus for any $x,y,z\in H$ and for all $a\in(xoy)oz=(xoz)oy$,

$\mu(a) \geq \min\{\inf_{b\in(((xoz)oy)oy)o(0oy)}\mu(b), \mu(0)\}$.

This implies, $\mu(a) \geq \inf_{b\in((xoz)oy)oy}\mu(b)$, for all $a\in(xoy)oz$
that is, $\inf_{a\in(xoy)oz} \geq \inf_{b\in((xoy)oy)oz}\mu(b)$ (since $((xoz)oy)oy=((xoy)oy)oz$)
Therefore for any $x,y,z\in H$ and for all $t\in xo(yo(yox))$,
.
Which gives, $\mu(t) \geq \min\{\inf_{b\in((xoy)oy)oz}\mu(b), \mu(z)\}$.
Hence μ is a fuzzy hyper BCK-implicative ideal of H.

5. Conclusion

Every (fuzzy) (weak, strong, reflexive) hyper BCK-positive implicative ideal of a hyper BCK-algebra H is a (fuzzy) (weak, strong, reflexive) hyper BCK-ideal of H. The product of two fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideals is also a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal. A fuzzy set μ in a hyper BCK-algebra H is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ is both a fuzzy (weak, strong, reflexive) hyper BCK-positive implicative ideal and a fuzzy (weak, strong, reflexive) hyper BCK-commutative ideal of H.

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