ISSN (Online) : 0974-5645
ISSN (Print) : 0974-6846
Indian Journal of Science and Technology, Vol 8(S3), 154-158, February 2015 DOI: 10.17485/ijst/2015/v8iS3/62247

# Fixed Point Theorems and Generalizations of Dislocated Metric Spaces

**P. Sumati Kumari[1*], Ch. V. Ramana[2], Kastriot Zoto[3] and D. Panthi[4]**

[1]Department of Mathematics, K L University, Green Fields, A.P., India; mumy143143143@gmail.com
[2]Department of Mathematics, Vignan University, Andhra Pradesh, India
[3]University of Gjirokastra, Gjirokastra, Albania
[4]Department of Mathematics, Valmeeki Campus, Nepal Sanskrit University, Nepal

## Abstract

In this note we discuss some topological properties and generalizations of dislocated metric space and establish some fixed point theorems. **Mathematics Subject Classification:** 47H10, 54H25.

**Keywords:** Fixed Point, Ld-Closed, Ld-Convergent, Ld-Limit, Ld-Metric, Ld-Open, Rd-Closed, Rd-Convergent, Rd-Limit, Rd-Metric, , Rd-Open

## 1. Introduction

The term *Distance Function* is customarily attributed to the metric associated with a metric space. However the study of the notion of a distance function on a set goes back to Frechet and Frink[1, 2]. In 2000, Hitzler and Seda[3] proved fixed point theorems in complete dislocated metric spaces. Their theorems generalizes Banach contraction principle to dislocated metric spaces. Sarma[4] introduced the topological aspects and completion of dislocated metric spaces and proved a coincidence theorem which generalizes Mathew's theorem. Moreover he proved a dislocated metric version of Seghal's fixed point theorem which ultimately implies to satisfy conditions analogous to those of Banach, Kannan, Bianchinl, Reitch and Rakotch. Ahmed[5] introduced the concept of left (right) dislocated metric spaces and proved some fixed point theorems. In recent years many authors prove fixed point theorems in dislocated and dislocated quasi metric spaces(see for examples[6-9]. In this paper we introduce topological aspects of left (right) dislocated metric spaces which is different from Ahmed's conditions and prove some fixed point theorems.

* *Author for correspondence*

## 2. Preliminaries and some Topological Properties

If $X$ is a set, any mapping $d$:$X \times X \to$R is called a distance function. In order that we are able to derive some meaningful results that are consistent with the natural laws of distance that are familiar, it is but natural to impose some natural restrictions on a distance function. The following are some of the most reasonable conditions for a distance function $d$ on a set $X$, which we consider in this note.

**Definition 1.** *Let X be a nomempty set. A distance on X is a map* $d$:$X \times X \to [0, \infty)$. *A pair (X, d) is known distance space if d satisfies the following conditions*

**($d_1$)** $d(x, x) = 0$

**($d_2$)** $d(x, y) = d(y, x) = 0 \Rightarrow x=y$

**($d_3$)** $d(x, y)= d(y, x)$

**($d_4$)** $d(x, z)\leq d(x, y)+d(y, z)$

If $d$ satisfies $(d_1)$-$(d_4)$ then it is called a metric on X. If $d$ satisfies $(d_2)$-$(d_4)$ then it is called dislocated metric on X.

**Definition 2.** *A distance function d is called right dislocated metric (rd-metric)or simply left dislocated metric (ld-*

*metric) iff it satisfies* $(d_4)$ *and the condition* $d(x, y) = 0 \Rightarrow x = y \ \forall \ x, y \in X$ *(resp.* $d(y, x) = 0 \Rightarrow y = x$*)*
It is clear that any dislocated metric is rd-metric (resp.ld-metric) but the converse need not be true.
If $x \in X$ and $\in > 0$ the set $\mathbb{B}^r_\in(x) = \{y : d(x, y) < \in\}$ (resp. $\mathbb{B}^l_\in(x) = \{y : d(y, x) < \in\}$) called the right open ball (resp. left open ball)
<u>Notation:</u> $V^r_\in(x) = \{x\} \cup \mathbb{B}^r_\in(x)$ (resp. $V^l_\in(x) = \{x\} \cup \mathbb{B}^l_\in(x)$)

**Proposition 1.** $\mathfrak{B} = \{V^r_\in(x)/x \in X\}$ (resp. $\{V^l_\in(x)/x \in X\}$) *is a basis for a topology on X.*
*Proof.* Clearly each $x \in X \ \exists \ V^r_\in(x) \ni x \in V^r_\in(x)$
If $y$ is a point of $V^r_\in(x)$ then there is a basis element $V^r_\delta(y)$ that contained in $V^r_\in(x)$
$y \in V^r_\in(x) = \mathbb{B}^r_\in(x) \cup \{x\} \Rightarrow d(x, y) < \in$
choose $\delta = \in - d(x, y)$ then $V^r_\in(y) \subset V^r_\in(x)$
$z \in V^r_\delta(x) \Rightarrow d(y, z) < \delta = \in - d(x, y) \Rightarrow d(x, z) < \in \Rightarrow z \in V^r_\in(x)$
Remark: Let $\mathfrak{J}^r_l$ (resp. $\mathfrak{J}^r_l$) be the topology induced by rd-metric (ld-metric)

**Proposition 2.** *If* $x \in X$*, the condition* $\{V^r_\in(x)/x \in X\}$ *(resp.* $\{V^l_\in(x)/x \in X\}$*) is an open base at x , hence* $\mathfrak{J}^r_d$ *(resp.* $\mathfrak{J}^r_l$ *) is first countable.*

**Definition 3.** *A sequence* $\{x_n\} \subset X$ *is right convergent (resp. ld- convergent) iff there exists a point* $x \in X$ *such that* lim $d(x, x_n) = 0$ *(resp.*.lim $d(x_n, x) = 0$*). In this case x is said to be the rd- limit (ld-limit) of* $\{x_n\}$
<u>Notation:</u> For $A \subset X$ we write $D_r(A) = \{x \in X/x$ is a right limit of $\{x_n\} \in A\}$Similary $D_l(A)$

**Definition 4.** *V is rd-open (resp.ld-open) iff* $\forall x \in V \ \exists \in > 0 \ni V^r_\in(x) \subset V$ *(resp.* $V^l_\in(x) \subset V$*)*

**Proposition 3.** *Let* $A \subset X$ *then* $x \in D_l(A) \Leftrightarrow$ *for every* $\in > 0 \quad \mathbb{B}^r_\in \cap A \neq \phi$ *(resp.* $\mathbb{B}^l_\in \cap A \neq \phi$*)*
Proof : Routine.
In[10] the authors introduced Dislocated symmetric spaces, highlighted some convergence axioms and proved some implications and non-implications among them. So we pick some convergence axioms from that article and prove some main theorems.
$(C_1)$ for a sequence and $\{x_n\} \in X, x, y \in X, \lim_{n\to\infty} d(x_n, x) = 0$ *and* $\lim_{n\to\infty} d(x_n, y) = 0$ *imply* $x = y$.
$(C_2)$ for a sequence $\{x_n\} \in X, x, y \in X, \lim_{n\to\infty} d(x, x_n) = 0$ *and* $\lim_{n\to\infty} d(y, x_n) = 0$ *imply* $x = y$.

**Definition 5.** *A sequence* $\{x_n\} \in X$ *is a Cauchy sequence if for every* $\in > 0$ *there corresponds a positive integer* $N_0$ *such that* $d(x_n, x_m) < \in$ *whenever* $n, m \geq N_0$.

**Proposition 4.** *Let (X, d) be a rd-metric space (resp.ld-metric space) X–A is rd-open (resp.ld-open) iff A is rd-closed (resp.ld-closed)*
*Proof.* Let (X, *d*) be a rd- metric space. A is rd-closed.
Claim : X-A is rd-open.
For $x \in X-A \ \exists r > 0 \ni V^r_\in(x) \subseteq X-A$. Suppose for some $x \in X-A$ this does not hold, then $\forall n \geq 1, \ V^r_{\frac{1}{n}} \subsetneq X-A$.
So $\exists \ x_n \in V^r_{\frac{1}{n}} \cap A \Rightarrow d(x, xn) < \frac{1}{n}$ x $d(x,xn)<1/n$. Since A is rd closed, x belongs to A.
which is a contradiction.
Conversely, suppose that X – A is rd-open.
Claim : A is rd-closed.
Suppose $x \in x, x_n \in A$ & $\lim_{n\to\infty} d(x, x_n) = 0$.
Let $x \notin X. \Rightarrow \exists \in 0 \ni V^r_\in(x) \subseteq X - A$. For this $\in \ \exists N_\in \ni d(x, x_n) < \in \forall n \geq N_\in \Rightarrow x_n \notin$ A which is contradiction.

**Definition 6.** *The pair (X, d) is said to be rd-complete (resp. ld- complete) if every cauchy sequence is rd-convergent (resp.ld-convergent)*

<u>**Remark:**</u> rd-limit (resp.ld-limit) of a sequence in (X, d) is unique.

**Proposition 5.** $A \subset X$ *is rd-closed (resp.ld-closed) iff A contains all its limit points.*
*Proof.* Suppose that A is rd-closed.
$\Rightarrow$ X – A is rd-open. Let $\{x_n\}$be a sequence in A $\ni d(x, x_n) < \in \forall n \geq N_\in$ ....(1)
Claim: $x \in A$
Suppose $x \notin A \Rightarrow x \in X - A$. Since X - A is rd-open $\exists \in > 0 \ni V^r_\in(x) \subseteq X - A$
From (1) $x_n \in V^r_\in(x) \subseteq$ X – A. Which is a contradiction.
Conversely Suppose that X - A is not open.
$\exists \ x \in X - A \ni$ for every $\in > 0, \mathbb{B}_\in(x) \nsubseteq X - A$ i.e $\mathbb{B}_\in(x) \cap A \neq \phi$
So for every positive integer n, $\exists x_n \in \mathbb{B}_{1/n}(x) \cap A \Rightarrow d(x, x_n) < 1/n$
$x_n \in A$. lim $d(x, x_n) = 0$. x is a limit point of A and $x \notin A$ which is a contradiction
**Definition 7.** *Let (X, d) be a rd-metric (resp.ld-metric) space and* $f{:}X \to X$ *f is said to be right continuous or simply (rd-continuous) (resp.ld-continuous) at x iff* $\forall \in > 0 \ \exists \ \delta > 0 \ni f(V^r_\delta(x)) \subseteq V^r_\in(f(x))$ *(resp.*$f(V^l_\delta(x)) \subseteq V^l_\in(f(x))$*)*

**Definition 8.** *Let (X, d) be a rd-metric space and $f:X\to X$. If there is a number $0 < \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha\, d(x, y)$ $\forall x, y \in X$ then f is called a contraction.*

**Proposition 6.** *Let (X, d) be a rd-metric (resp ld-metric ) space and f is a contraction then f is rd-continuous (resp. ld-continuous).*

*Proof.* *Suppose f is not rd- continuous at $x \in X$.*

$\exists \in\ > 0\ \forall n > 0; f(V_{\frac{1}{n}}^{r}(x)) \nsubseteq\ _{\in}\ (f(x)$

$\Rightarrow \exists(x_n) \in X \ni d(x, x_n) < 1/n$ *but* $d(f(x), f(x_n)) \geq \in$

*Since* $d(f(x), f(x_n)) \leq c\, d(x\, x_n) < 1/n$; $\lim d(f(x), f(x_n)) = 0$

*So,* $\exists N \ni d(f(x), f(x_n)) < \in \forall\ n \geq N$. *This is a contradiction*

*Hence f is rd-continuous at x.*

*Similarly f is ld- continuous.*

**Proposition 7.** *If $d(x, y)<\delta \Rightarrow d(y, x), \delta$ then $\mathfrak{J}_d^r$ coincides with $\mathfrak{J}_d^l$ and this topology satisfies Housdorff property.*

*Proof.* if $d(x, y) < \delta \Rightarrow d(y, x) < \delta$ then,

$d(x, y) < \delta \Leftrightarrow d(y, x) < \delta$

So that $\mathbb{B}_\delta^r(x) = \mathbb{B}_\delta^l(x)$ Then the right topology $\mathfrak{J}_d^r$ generated by $d$ coincides with the left topology.

Now suppose, $x \neq y$, then $d(x, y) > 0$. Let $0 < \delta < d(x,y)/2$

If $\exists z \in V_\delta(x) \cap V_\delta(y)$ then $x \neq z \neq y$ and

$d(x, z) < \delta$ and $d(y, z)\ \delta \Rightarrow d(z, y) < \delta$

Hence $d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \delta < 2\delta < d(x, y)$:

**Proposition 8.** *$\mathfrak{J}_d^r$ (resp. $\mathfrak{J}_d^l$ ) is $T_1$ space $\Leftrightarrow$ d satisfies $d(y, x)>0$ (resp. $d(x, y)>0$)*

*Proof. Assume that $\mathfrak{J}_d^r$ is $T_1$ space.*

*Let $x \in X$, $y \in X$ and $x \neq y$. Since $\mathfrak{J}_d^r$ d is open, $X - \{x\}$ is open.*

$x \neq y \Rightarrow y \in X - x.$

$\Rightarrow \exists \delta > 0 \ni V_\delta^r \subseteq X - \{x\}$

$\Rightarrow\ x \notin V_\delta^r(y)$

$\Rightarrow d(y, x) \geq \delta > 0.$

Conversely, suppose that the condition holds. We show that $X - \{x\}$ is open $\forall x \in X$

Let $y \in X - \{x\}$ since $y \neq x$, $d(y, x) > 0$

$z \in V_\in^r(y) \Rightarrow d(y, z) < \in \Rightarrow z \neq x$

$z \in X - \{x\} \Rightarrow V_\in^r(y) \subseteq (y)\, X - \{x\}$

Hence $X - \{x\}$ is closed. Since this holds $\forall x \in X$, $(X, \mathfrak{J}^r)$ is $T_1$ space.

## 3. Main Results

**Theorem 1.** *Let (X, d) be a right complete (left complete) dislocated metric space with C(2)and let f be a contraction mapping on X. Then f has a unique fixed point.*

*Proof.* let x, y be any two points of X. Since f is a contraction on X there exists a real number $\alpha$ with $0 \leq \alpha < 1$ such that

$d(f(x), f(y)) \leq \alpha d(x, y)$

In fact, $d(f^2(x), f^2(y)) \leq \alpha d(f(x), f(y)) \leq \alpha^2 d(x, y)$

Then by induction, for any positive integer n,

$d(f^n(x)), f^n(y)) \leq \alpha^n d(x, y) \forall x, y \in X$

Now let, $x_0$ be any point of X, set for $n \geq 0$

$x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0) \ldots, x_{n+1} = f(x_n)$

$x_n = f(x_{n-1}) = \ldots = f^n(x_0)$

Let $m, n(m > n)$ be any positive integer $\geq 1$, then we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &= d(f^n(x_0), f^n(x_1)) + \ldots + d(f^{m-1}(x_0), f^{m-1}(x_1)) \\ &\leq (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}) d(x_0, x_1) \\ &= \alpha^n(1 + \alpha + \ldots + \alpha^{m-n-1}) d(x_0, x_1) \\ &< \frac{\alpha^n}{1-\alpha} d(x_0, x_1) \end{aligned}$$

Since $0 \leq \alpha < 1$, $\{x_n\}$ is a cauchy sequence in (X, d), hence right convergent

i.e $\lim d(x, x_n) \to 0$ ...(1)

Since 'f' is continuous, $\lim d(f(x), f(x_n)) \to 0$. ...(2)

from (1) and (2) $f(x) = x$

**Uniqueness:**

$d(x, z) = d(f(x), f(z)) < \alpha d(x, z) \Rightarrow d(x, z) = 0 \Rightarrow x = z$

**Theorem 2.** *Let (X, d) be a right complete d-metric space and $f: X \to X$ be continuous. Assume that there exists a number 'a' ; $0 \leq a < 1/2$ such that for each $x, y \in X$ $d(f(x), f(y)) \leq a[d(x, f(x) + d(y, f(y))]$ then f has a unique fixed point.*

*Proof.* For any $x \in X$

$d(f(x), f^2(x)) \leq a[d(x, f(x) + d(f(x), f^2(x))]$

$\Rightarrow d(f(x), f^2(x)) \leq \frac{a}{1-a}\, d(x, f(x))$

$\Rightarrow d(f(x), f^2(x)) \leq \beta d(x, f(x))$, where $\beta = \frac{a}{1-a}$.

then by induction, let $m, n(m > n)$ be any positive integer $\geq 1$

$d(f^n(x)), f^m(y)) \leq \beta^n(1 + \beta + \ldots + \beta^{m-n-1})\, d(x, f(x))$

$\leq \frac{\beta^n}{1-\beta}\, d(x, f(x))$

Hence, $\{f^n(x)\}$ is Cauchy sequence in $(X, d)$.

Therefore right convergent $(x, f^n(x)) \to 0$....(1)

Since f is continuous $d(f(x), f^n(x)) \to 0$...(2)

from (1) and (2) $x = f(x)$.

**Uniqueness:**

$$\begin{aligned} d(x, z) &= d(f(x), f(z)) \\ &\leq a[d(x, f(x)) + d(z, f(z)] \\ &= a[d(x, x) + d(z, z)] \ldots (3) \end{aligned}$$

But,

$d(x, x) \leq a[d(x, f(x) + d(x, f(x))]$
$\quad = 2ad(x, f(x))$
$\quad = 2ad(x, x)$

$d(x, x) = 0$; Similarly $d(z, z) = 0 \Rightarrow x = z$ (from (3)).

**Theorem 3.** *There exists a number 'a', $0 < a < \frac{1}{2}$ such that for each $x, y \in X$*

*$d(f(x), f(y)) \leq a[d(x,f(y)) + d(y, f(x))]$*

*Proof.*

$d(f(x); f^2(x)) \leq a[d(x, f^2(x) + d(f(x), f(x))]$...(1)

Now

$d(f(x), f(x)) \leq a[d(x, f(x) + d(x, f(x))]$
$\quad = 2ad(x, f(x))$

from (1)

$d(f(x), f^2(x)) \leq ad(x, f(x) + ad(f(x), f^2(x)) + 2a^2 d(x, f(x))$

$\Rightarrow d(f(x), f^2(x)) \leq \frac{a+2a^2}{1-a}\ d(x, f(x))$

$\Rightarrow d(f(x), f^2(x)) \leq \beta d(x, f(x))$ where $\beta = \frac{a+2a^2}{1-a}$

**Uniqueness:**

$d(x, z) = d(f(x), f(z))$
$\quad \leq a[d(x, x) + d(z, z)]$

$d(x, x) = d(f(x), f(x))$
$\quad \leq 2ad(x, x)$

$d(x, x) = 0$, similarly $d(z, z) = 0$. Hence $x = z$

**Theorem 4.** *Let $(X, d)$ be a right complete d-metric space and $f: X \to X$ be continuous. Assume that there exists a number h, $0 \leq h < 1$ such that for each $x, y \in X$*

*$d(f(x), f(y)) \leq h \max [d(x, f(x), d(y, f(y)))]$*

*then f has a unique fixed point.*

*Proof.* For each $x \in X$,

$d(f(x), f^2(x)) \leq hd(x, f(x)$ : *where* $0 \leq h < 1$

therefore $\{f^n(x)\}$ is a Cauchy sequence. So $f(x) = x$

**Uniqueness:**

$d(x, z) = d(f(x), f(z))$
$\quad \leq h \max [d(x, x), d(z, z)]$

$d(x, x) = d(f(x), d(fx))$
$\quad \leq h \max [d(x, x), d(x, x)]$
$\quad = hd(x, x)$

$\Rightarrow d(x, x) = 0$, Similarly $d(z, z) = 0$. Hence $d(x, z) \Rightarrow x = z$

**Theorem 5.** *Let $(X, d)$ be a right complete d-metric space and $f: X \to X$ be continuous. Assume that there exists non negative number a, b, c satisfying $a + b + c < 1$ such that for each $x, y \in X$*

*$d(f(x), f(y)) \leq ad(x, f(x) + bd(y, f(y) + cd(x, y))$*

*then f has a unique fixed point.*

*Proof.* For each $x \in X$

$d(f(x), f^2(x)) \leq ad(x, f(x)) + bd(f(x), f^2(x)) + cd(x, f(x))$

$\Rightarrow d(f(x), f^2(x)) \leq \frac{a+c}{1-b}\ d(x, f(x))$

$\Rightarrow d(f(x), f^2(x)) \leq \beta d(x, f(x)), 0 \leq \beta < 1$

If we follow the same steps as in above theorem, we get $f(x) = x$

**Uniqueness:**

$d(x, z) = d(f(x), f(z)$
$\quad \leq a\ d(x, f(x)) + b\ d(z, f(z)) + c\ d(x, z)$
$\quad = a\ d(x, x) + b\ d(z, z) + c\ d(x, z)$...(1)

$d(x, x) = d(f(x), f(x))$
$\quad \leq a\ d(x, f(x)) + b\ d(x, f(x)) + c\ d(x, x)$
$\quad \leq (a + b + c)d(x, x)$

Since, $(a + b + c) < 1 \Rightarrow d(x, x) = 0$. Similarly $d(z, z) = 0$

From (1), $d(x, z) \leq c\ d(x, z)$

$\Rightarrow (1 - c)\ d(x, z) \leq 0 \Rightarrow d(x, z) = 0$. Hence, $x = z$.

**Theorem 6.** *Let X be a non-empty set. Define (i) $D(x, y) = d(x, y) + d(y, x)\ \forall x, y \in X$. Then D is dislocated metric iff d is both rd-metric and ld-metric.*

*(ii)$\mathfrak{J}_D = \mathfrak{J}_d^r \cap \mathfrak{J}_d^l$ where $\mathfrak{J}_D$ is topology induced by D.*

Proof of (i) Clearly $D(x, y) \geq 0$ as $d(x, y) \geq 0$ and $d(y, x) \geq 0$

$D(x, y) = D(y, x) = 0 \Rightarrow x = y$. Triangle inequality holds.

Hence D is a dislocated metric space.

Proof of (ii)

$$\mathfrak{J}_D = \mathfrak{J}_D^r \cap \mathfrak{J}_D^l$$

Since $\mathbb{B}_{D\in}\ (x) \subseteq \mathbb{B}^r_{d\in}\ (x) \cap \mathbb{B}^l_{d\in}\ (x)$

$$\mathbb{B}^r_{d_{\frac{\delta}{2}}}\ (x) \cap \mathbb{B}^l_{d_{\frac{\delta}{2}}}\ (x) \subseteq \mathbb{B}_{D\in}\ (x)$$

*where* $\mathbb{B}_{D\in}\ (x) = \{y/D(x, y) < \in\}$

$$\mathbb{B}^r_{d\in}\ (x) = \{y/d(x, y) < \in\}$$

$$\mathbb{B}^l_{d\in}\ (x) = \{y/d(y, x) < \in\}$$

$$\mathbb{B}_{D\in}\ (x) \subseteq \mathbb{B}^r_{d\in}\ (x) \cap \mathbb{B}^l_{d\in}\ (x)$$

*Let* $y \in \mathbb{B}_{D\in}\ (x) \Rightarrow D(x, y) < \in$
$\quad \Rightarrow d(x, y) + d(y, x) < \in$
$\quad \Rightarrow d(x, y) < \in$ *and* $d(y, x) < \in$
$\quad \Rightarrow y \in \mathbb{B}^r_{d\in}\ (x)$ *and* $y \in \mathbb{B}^l_{d\in}\ (x)$

$y \in \mathbb{B}^r_{d\in}\ (x) \cap \mathbb{B}^l_{d\in}\ (x)$

$\mathbb{B}^r_{d_{\frac{\delta}{2}}}\ (x) \cap \mathbb{B}^l_{d_{\frac{\delta}{2}}}\ (x) \subseteq \mathbb{B}_{D\in}\ (x)$

$y \in \mathbb{B}^r_{d_{\frac{\delta}{2}}}\ (x) \cap \mathbb{B}^l_{d_{\frac{\delta}{2}}}\ (x) \subseteq \mathbb{B}_{D\in}\ (x)$

$d(x, y) < \delta/2$
$d(y, x) < (\delta/2)$
$D(x, y) < \delta$
$y \in \mathbb{B}_{D\delta}(x)$

**Theorem 7.** *A is closed in $\mathfrak{J}_D$ iff A is both right closed and left closed.*
*Proof.* Claim: A is closed in $\mathfrak{J}_D$
i.e. for every $\in > 0$, there exists a positive integer $n_\in \ni D(x_n, x) < \in \forall n \geq n_\in$
$\Rightarrow d(x_n, x) + d(x, x_n) < \in$
$\Rightarrow d(x_n, x) < \in$ and $d(x, x_n) < \in$
$\Rightarrow \lim d(x_n, x) = 0$ and $\lim d(x, x_n) = 0$
Since A is both right closed and left closed so $x \in$ A. Hence A is $\mathfrak{J}_D$ closed.
Assume that A is $\mathfrak{J}_D$ closed.
Claim : A is right closed and left closed
Let $\{x_n\}$ be a sequence in A and $x \in$ X such that $\lim d(x, x_n) = 0$ and
$\lim d(x_n, x) = 0$ i.e. for every $\delta > 0$ $\exists$ positive integers $n_{\delta 1}$ and $n_{\delta 2} \ni$
$d(x, x_n) < \delta/2$ and $d(x_n, x) < \delta/2$
For $n \geq n_{\delta 1}$ and $n \geq n_{\delta 2}$ respectively
$D(x_n, x) = d(x, x_n) + d(x_n, x) < \delta \Rightarrow D(x_n, x) < \delta \ \forall \ n \geq min\{n_{\delta 1}, n_{\delta 2}\}$
$\Rightarrow \lim D(x_n, x) = 0 \Rightarrow x \in$ A.

**Theorem 8**. *X is D-complete iff X is both right complete and left complete.*
*Proof.* Claim : *X* is *D*-complete
Assume that X is both right complete and left complete. Let $\{x_n\}$ be a cauchy sequence in X with respect to D. For every $\in > 0$ $\exists$ a positive integer $N_{\in 1} \ni D(x_n, x_m) < \in \forall n, m \geq N_{\in 1}$
$\Rightarrow d(x_n, x_m) + d(x_m, x_n) < \in_1$
$\Rightarrow d(x_n, x_m) < \in_1$ and $d(x_m, x_n) < \in_1 \ \forall n, m \geq N_{\in 1}$
$\Rightarrow min\{d(x_n, x_m), d(x_m, x_n)\} < \in_1$. Since X is both left and right complete, $\lim d(x, x_n) = 0$ and $\lim d(x_n, x) = 0$. Note that $\lim D(x_n, x) = 0$. Since, $\lim D(x_n, x) = \lim d(x, x_n) + \lim d(x_n, x) \rightarrow 0$.
Conversely, assume that X is D-complete.
We have to prove that X is both right and left complete.
Let $\{x_n\}$ be a right Cauchy-sequence in X.
$min\{d(x_n, x_m), d(x_m, x_n\} < \in/2$
For every $\in > 0$ $\exists$ a positive integer $N_1 \ni d(x_m, x_n) < \in/2$ $\forall m, n \geq N_1$. Similarly, $d(x_n, x_m) < \in/2 \ \forall m, n \geq N_2$.
Hence, $D(x_n, x_m) = d(x_n, x_m) + d(x_m, x_n) \forall m, n \geq N$ where $N = min\{N_1, N_2\}$.
So, $\{x_n\}$ is a D-Cauchy sequence.
since X id D-complete, $\lim D(x_n, x) = 0$ i.e. for every $\delta > 0$ $\exists$ a positive integer $N_\delta \ni D(x_n, x) < \delta \forall n \geq N_\delta$
$\Rightarrow d(x_n, x) + d(x, x_n) < \delta \ \forall n \geq N_\delta$
$\Rightarrow d(x_n, x) < \delta$ and $d(x, x_n) < \delta$
$\Rightarrow \lim d(x_n, x) = 0$ and $\lim d(x, x_n) = 0$
X is both right and left complete.

**Theorem 9 (Mathews Theorem)** *Let (X, d) be a complete dislocated metric space. f: X → X be a contraction then f has a unique fixed point.*
Since (X, d) is a complete dislocated metric space. If (X, d) is both right complete and left complete and *f*: *X* → *X* be a contraction then from Mathews Theorem *f* has a unique fixed point.

## 4. References


1. Frechet CF, Sur quelques points du calcul fonctionnel. Rendiconti del Circolo Matematico di Palermo. 1906; 22(1):1–72.
2. Frink CAH. Distance functions and the metrication problem. Bull Am Math Soc. 1937; 43.
3. Hitzler P, Generalized metrics and topology in logic programming semantics [PhD Thesis]. National University of Ireland (University College, Cork); 2001.
4. Sarma IR, Kumari PS. On dislocated metric spaces. Int J Math Arch. 2012; 3(1):72–7.
5. Ahamad MA, Zeyada FM and Hasan GF. Fixed point theorems in generalized types of dislocated metric spaces and its applications. Thai J Math. 2013; 11:67–73.
6. Kumari PS. On dislocated quasi metrics. Jour of Adv Stu in Top. 2012; 3(2):66–74.
7. Panthi D. Fixed Point results in cyclic contractions of generalized dislocated metric spaces. Annals of Pure and Applied Mathematics. 2014; 5(2):192–7.
8. Kumari et al. Common fixed point theorems on weakly compatible maps on dislocated metric spaces. Mathemat Sci. 2012; 6:71. doi:10.1186/2251-7456-6-71.
9. Zoto K, Hoxha E, Isufati A. Some new results in dislocated and dislocated quasi metric spaces. Appl Math Sci. 2012; 71:3519.
10. Sarma IR, Rao JM, Kumari PS, Panthi D. Convergence axioms on dislocated symmetric spaces. Abstr Appl Anal.