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# Anisotropic Constitutive Model for Predictive Analysis of Composite Laminates

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## Abstract

An anisotropic plastic constitutive model for fiber-reinforced composite material, is developed, which is simple and efficient to be implemented into computer program for a predictive analysis procedure of composite laminates. An anisotropic initial yield criterion, as well as work-hardening model and subsequent yield surface are established that includes the effects of different yield strengths in each material direction, and between tension and compression. The current model is implemented into a computer code, which is Predictive Analysis for Composite Structures (PACS). The accuracy and efficiency of the anisotropic plastic constitutive model and the computer program PACS are verified by solving a number of various fiber-reinforced composite laminates. The comparisons of the numerical results to the experimental and other numerical results available in the literature indicate the validity and efficiency of the developed model.

**Keywords:** Anisotropic Plasticity, Fiber Composite, Predictive Finite Element Analysis

## 1. Introduction

The high strength and stiffness-to-weight ratios of composite materials make these materials attractive for certain critical applications[1]. Composites have been increasingly used as structural materials in the space and aerospace industry, aircraft industry, automobile industry, and in various engineering fields.

Composite materials partly behave in a nonlinear fashion, although composite materials generally have been modeled as linear elastic material[2]. The nonlinearity of composite materials can be attributed to inherent material nonlinearity of individual constituents and to micromechanical failures such as fiber or matrix microcracking and interfacial debonding[3]. These phenomena may be described macroscopically within the framework of plasticity theory.

It is well known that the plasticity theory is capable of mathematically modeling the inelastic material behavior at macroscopic level[3]. However, the available plasticity constitutive models for isotropic materials are not sufficient to represent the nonlinear behavior of composite materials. On the other hand, several theoretical anisotropic plasticity constitutive models have been developed, these are generally for metal, or too complex to be implemented into a numerical computer code[4].

Hence, general response prediction of composite structures becomes possible by developing a realistic and comprehensive analysis procedure for general loading conditions[5]. Such an undertaking, among other considerations, requires very efficient constitutive model, which can predict realistically nonlinear material behavior[6]. It is attempted to develop an anisotropic plasticity constitutive model for fiber-reinforced composite laminates that is simple and efficient to be implemented into a computer program for a predictive analysis procedure of composite laminates.

## 2. Constitutive Model for Composite Materials

### 2.1 Elastic Constitutive Relationship

The most general stress-strain relationship within the linear elasticity can be written in tensor notation as

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \tag{1}$$

where $\sigma_{ij}$ is the stress vector, $D_{ijkl}$ the stress-strain matrix, and $\varepsilon_{kl}$ the strain vector. The planes of symmetry are

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aligned with material principal axes (1, 2, 3) in an lamina. With the contracted notation, the stress-strain relation for a lamina becomes.

$$\sigma_i = D_{ij}\varepsilon_j \tag{2}$$

## 2.2 Anisotropic Plasticity Theory

It should be mentioned that incremental theory of plasticity is used as the base of development for an anisotropic plasticity theory, since it more adequately reflects the progressive behavior of material. The following three fundamental elements of plasticity are developed for the anisotropic plasticity theory for fiber-reinforced composite laminates:

(a) An initial anisotropic yield surface that defines the elastic limit of material behavior in the stress space.
(b) An anisotropic hardening rule that specifies the evolution of subsequent yield surface under plastic deformation.
(c) The flow rule, clarifying the direction of the incremental plastic strain vector in strain space.

## 2.3 Anisotropic Yield Criterion

Considering the material characteristics of the fiber-reinforced composites, a general form of the anisotropic yield criterion for composite material can be the quadratic function given as

$$f = A_{ij}\left(\sigma_i - \alpha_i\right)\left(\sigma_j - \alpha_j\right) - \kappa^2 \quad \left(i, j = 1, 2, \ldots, 6\right) \tag{3}$$

where $\sigma_i$ is the current state of stress and where three material variables $A_{ij}$, $\alpha_i$ and $k$ represent the current state of plastic deformation. Anisotropic parameters $A_{ij}$ describe the current state of plastic anisotropy as represented by different yield stresses with respect to material directions. The translation vector $\alpha_i$ describes the current strength differentials between tensile and compressive yield stresses. The scalar parameter represents the effective size of yield surface that corresponds to the reference yield stress of material at a given point in loading history.

Composite material considered in this study is assumed to have three orthogonal symmetric planes (Figure 1), and the principal axes of anisotropy are assumed as reference axes. The yield function Eq. (3) can also be expressed in explicit form as

$$f = A_{ij}\sigma_i\sigma_j - A_i\sigma_i - K \tag{4}$$

where, $A_i = 2A_{ij}\alpha_j$ and $K = -A_{ij}\alpha_i\alpha_j + \kappa^2$.

The above yield function is more convenient for finding $A_{ij}$ and $A_i$ from experimental results. A physical interpretation of Eq. (4) can be made by uniaxial tension and compression tests in each direction, and by pure shear tests, respectively. The anisotropic material parameters in Eq. (4), which describe plastic anisotropy in material principal directions, are obtained except off-diagonal term of $A_{ij}$.

## 2.4 Evaluation of Interaction Term

In order to determine the remaining off-diagonal interaction term in the anisotropic parameter, either of any biaxial tests like tension-tension, tension-compression, or zero volumetric plastic strain assumption (i.e., the incompressibility) has been used. The bound restriction (stability condition) for an elliptic equation is used to determine interaction term. The yield surface of Eq. (4) is an ellipsoid in a three-dimensional stress space with $\sigma_1$, $\sigma_2$, and one of the shear stresses. Hence, the yield surface will be an ellipse in a plane of constant shear stress (Figure 2) if the

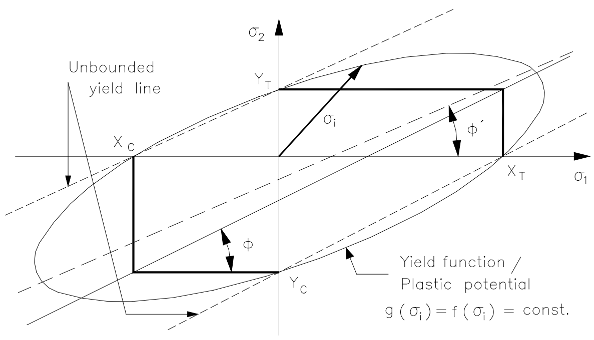


**Figure 1.** Assumed rotation of yield surface in a constant shear stress plane and plastic increment vector in the plastic potential.

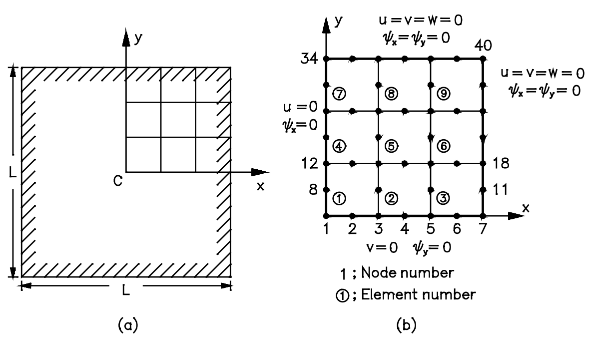


**Figure 2.** The geometry of clamped plate and finite element discretization.

following condition in terms of anisotropic parameters is satisfied:

$$A_{11}A_{22} - A_{12}^2 \;\rangle\; 0 \tag{5}$$

When an elliptic equation satisfies the stability condition, the rotation angle of the major axis of ellipse shows the following relation with the anisotropic material parameters of Eq. (4):

$$\tan 2\phi' = \frac{2A_{12}}{A_{11} - A_{22}} \tag{6}$$

where $\phi'$ is the rotation angle between the major axis of yield ellipse in Figure 1.

In the current model, the rotation of the major axis to the strength axis in 1-direction is assumed as

$$\tan \phi = \frac{Y_T + Y_C}{X_T + X_C} \tag{7}$$

where $\phi$ is the assumed rotation angle, and $X_T$, $X_C$, $Y_T$, and $Y_C$ are the tensile and compressive yield stresses in 1- and 2- directions, respectively. By substitution of Eq. (7) into Eq. (6), $A_{12}$ can be written as

$$A_{12} = \frac{\left(X_T + X_C\right)\left(Y_T + Y_C\right)}{\left(X_T + X_C\right)^2 - \left(Y_T + Y_C\right)^2}\left(A_{11} - A_{22}\right) \tag{8a}$$

or,

$$A_{12} = \frac{\left(X_T + X_C\right)\left(Y_T + Y_C\right)}{\left(X_T + X_C\right)^2 - \left(Y_T + Y_C\right)^2}\left(\frac{K}{X_T X_C} - \frac{K}{Y_T Y_C}\right) \tag{8b}$$

The new interaction terms given by Eq. (8) automatically satisfy the stability condition, Eq. (5).

## 2.5 Anisotropic Work-Hardening Rule

An attempt is given in here to determine the anisotropic material parameters $A_{ij}$ in the principal material directions with varying yield strengths. The anisotropic parameters for the subsequent yield criterion at any state of plastic deformation can be obtained as

$$A_{ij}(\bar{\sigma}) = \frac{\bar{\sigma}^2}{X_i^2} = \frac{\bar{\sigma}^2}{\left[\frac{E_i^p}{H'}(\bar{\sigma}^2 - \bar{\sigma}_0^2) + X_{0i}^2\right]} \tag{9}$$

where $\bar{\sigma}$ is effective stress and $E^p$ is plastic modulus, $H'$ is hardening modulus.

The incremental constitutive relationship for an elastic-plastic material is formulated by using the consistency condition. With the initial and subsequent yield criteria, as well as the incremental elastic-plastic constitutive relationship, we can describe the nonlinear material behavior. The incremental elastic-plastic constitutive relationship is presented as

$$d\sigma_i = D_{ij}^{ep} d\varepsilon_j \tag{10}$$

where $d\sigma_i$ and $d\varepsilon_j$ are the incremental stress and strain vectors, respectively, $D_{ij}^{ep}$ is the elastic-plastic material stiffness tensor defined as

$$D_{ij}^{ep} = D_{ij}^{e} - \frac{D_{ik}^{e}\dfrac{\partial f}{\partial \sigma_k}\dfrac{\partial \bar{\sigma}}{\partial \sigma_i}D_{lj}^{e}}{\dfrac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}^p} + \dfrac{\partial f}{\partial \sigma_m}D_{mn}^{e}\dfrac{\partial \bar{\sigma}}{\partial \sigma_n}} \tag{11}$$

## 3. Numerical Illustrative Examples

In order to verify the validity of the problem formulation and the efficiency of numerical implementation, several example problems are analyzed by using the computer program which is a Predictive Analysis for Composite Structures (PACS) [6]. Numerical results are compared with various experimental and other numerical results.

A clamped square plate is considered under uniformly distributed and concentrated load at center, respectively. A clamped square plate under uniformly distributed loading is considered. As shown in Figure 2, only a quarter of the plate has been discretized because of symmetry considerations. Eight-noded isoparametric elements with four-point (2x2) Gaussian integration in the plane are used, and eight equal thickness layers are taken through thickness direction.

The plate has been solved with both isotropic and anisotropic materials; material properties are given in Table 1.

**Table 1.** Material constants used in clamped plate (Unit: $MN/m^2$)

| | Isotropic material | Anisotropic material |
|---|---|---|
| Elastic modulus | $E_1$ = 30,000.0, $E_2$ = 30,000.0 | $E_1$ = 30,000.0, $E_2$ = 30,000.0 |
| Poisson's ratio | $\nu_{12} = \nu_{21}$ = 0.3 | $\nu_{12} = \nu_{21}$ = 0.3 |
| Shear modulus | $G_{12} = G_{23} = G_{31}$ = 11,540.0 | $G_{12} = G_{23} = G_{31}$ = 11,540.0 |
| Yield strength | $\sigma_1 = \sigma_2$ = 30.0, $\tau_{12} = \tau_{23} = \tau_{31}$ = 17.32 | $\sigma_1$ = 30.0, $\sigma_2$ = 40.0, $\tau_{12}$ = 20.2, $\tau_{23} = \tau_{31}$ = 17.32 |
| Plastic modulus | $E_p$ = 300.0, $G_p$ = 100.0 | $E_p$ = 300.0, $G_p$ = 100.0 |

To verify the validity of the computer program, the results are compared with other numerical ones[4]. Figure 3 shows load-displacement at center of the plate. With the close agreement, the capability of the program PACS is presumed to be valid.

The PACS is also used to predict nonlinear anisotropic material behavior of the fiber-reinforced composite laminates. The results are compared with experimental and other numerical results. Tensile stress-strain curve for $[0°/90°]_s$ cross-ply Boron/Epoxy laminate is shown in Figure 4. The results of PACS are rather close to the experimental results of [1] than other numerical ones[2].

Here, the simple supported plate on four sides of the plate with geometry of plate and finite element discretization given in Figure 2 is analyzed. The purpose of the analysis is to see effects of different boundary conditions on the plastic behavior of the plate. In order to compare with the clamped plate, the same geometry as the clamped plate is used with a uniformly loaded case. The comparison is performed only on isotropic material. The development of plastic regions for both clamped and simple supported plates are presented in Figure 5.

Yielding starts at the corners in the simple supported plate, while starting in the middle of the sides of the clamped plate. Next, the center of both plates yields. Then the plastic regions propagate from the vicinity of the corners and the plate centers. The pattern of the plastic region growth agrees with that of [7].

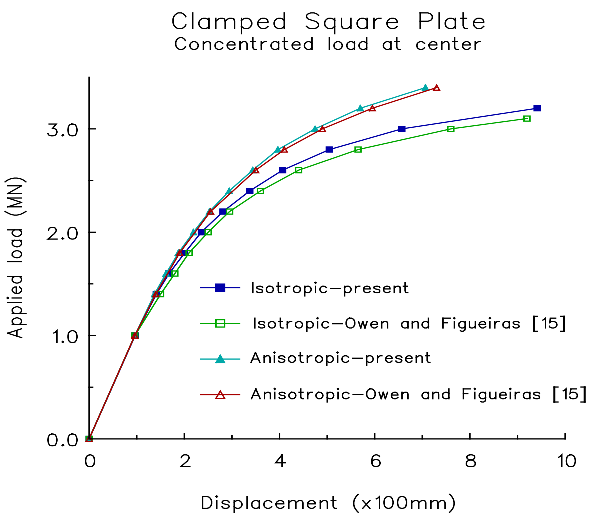


**Figure 3.** Load-displacement at center of the plate.

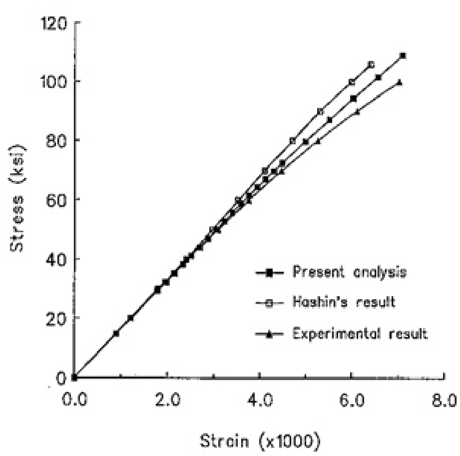


**Figure 4.** Tensile stress-strain curve.

## 4. Conclusion

The development and use of an analytical procedure which has capability of predicting the progressive material behavior of structures, named as a predictive analysis procedure in here, is developed for more accurate assessment of structural safety and efficiency of composite structures. A quadratic anisotropic yield criterion in stress

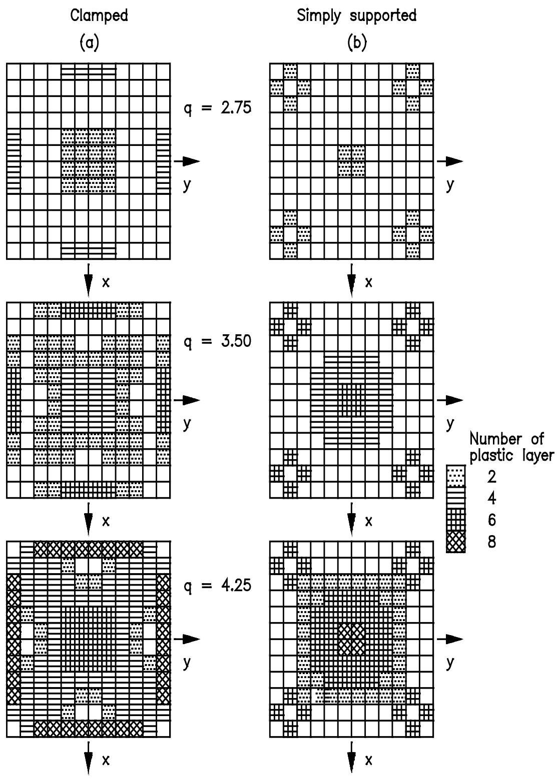


**Figure 5.** Development of plastic region for (a) clamped and (b) simple supported plate with uniform load.

space is developed for general use with unidirectional and bidirectional composite lamina. The developed anisotropic work-hardening model allows for a nonproportional change of the yield values so that the subsequent yield surface can be a distorted shape. The predictive finite element analysis procedure is developed for accurate and efficient analysis in predicting progressive behavior of composite structures. As a result, a computer code, PACS (Predictive Analysis of Composite Structures) is developed, which adopts the abovementioned anisotropic plasticity model.

The accuracy and efficiency of the computer code PACS are verified with various benchmark problems. Numerical predictions of the computer program PACS compare very well with available experimental, analytical and other numerical results. Comparisons illustrate the capability of the constitutive model and the computer program PACS. The developed constitutive and predictive analysis model predicts progressive nonlinear behavior from the beginning of loading, in plane and through thickness direction of composite laminates. The capability of PACS to predict nonlinear material behavior of composites can be used as a helpful device in parametric studies for design of fiber-reinforced laminates.

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