

Selective Banach Vector Fields

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Abstract

In this paper, the concept of tangent space for finite product of C^μ μ -selective Banach manifolds is introduced. Using this notion, the concept of differentiation of the mappings $f: M_0 \rightarrow N_0$ is extended to the differentiation of the mappings $g: M_1 \times M_2 \rightarrow N_1 \times N_2$, where M_i and N_i are μ_i -selective and γ_i -selective Banach manifolds for $i \in \{0, 1, 2\}$, respectively. Moreover, the notions of vector field and tensor field over μ -selective Banach manifolds are established.

Keywords: Observer, Selective Banach Manifold, Tangent Space, Tensor Field

1. Introduction

The problem of finding a mathematical model that can change the signature of a metric on a manifold was an interesting topic for both physicists and mathematicians^{1,2,7}. In 2004, it was highly expected to be solved due to introducing the mathematical model of one dimensional observer. Then, considering the concept of the observer over an n dimensional manifold, the notion of selective manifold was introduced. In fact, it was a realistic approach to the problem of unity in physics³. In 2009, the concept of a multi-dimensional observer was presented, and by using this notion, a new version of Tychonoff Theorem was proved, and an index for topological entropy was given⁴. In 2011, the notion of synchronization for continuous time dynamical systems from the observer viewpoint was established⁵. This notion is a generalization of synchronization, and it is proved that the future of the points of the sets in which two dynamical systems are relative probability synchronized are the same up to homomorphism determined by a relative probability synchronization⁶. Then, the idea of relative metric spaces as a mathematical model consistent with physical phenomena was considered.

Recently, the concept of the selective manifold over a Banach space was introduced, and its properties were studied. Also, the concept of the α -level differentiation

of the mappings between selective Banach manifolds was presented, and a new version of chain rule theorem was proved for the mappings between selective Banach manifolds. In addition, the notion of the tangent space of a selective Banach manifold at a given point was presented.

In this paper, the concept of tangent space for a finite product of selective Banach manifolds is defined, and using it, the notion of the α -level differentiation of mappings between such selective Banach manifolds is presented. In the end, concepts of vector field and tensor field are introduced. In the rest of this section, we provide some preliminaries on observers and selective Banach manifolds.

Let M be a set, and J be an index set composed of finite elements. A n dimensional observer on M is a mapping $\mu: M \rightarrow \prod_{j \in J} I_j$, where $I_j = [0,1]$ for every $j \in J$. Let $\mu: M \rightarrow \prod_{j \in J} I_j$ and $\eta: M \rightarrow \prod_{j \in J} I_j$ be two n dimensional observers on M . We write $\eta \subseteq \mu$ if $\eta_j(x) \leq \mu_j(x)$ for all $x \in M$ and $j \in J$. Moreover, we write $\eta \subset \mu$ if $\eta \subseteq \mu$ and for every $x \in M$, there exists $j \in J$ $\eta_j(x) < \mu_j(x)$. The n dimensional observers $\mu \cap \eta$ over M are defined as

$$(\mu \cup \eta)_j(x) = \sup\{\mu_j(x), \eta_j(x)\};$$

$$(\mu \cap \eta)_j(x) = \inf\{\mu_j(x), \eta_j(x)\}.$$

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Let, τ_μ be a collection of subsets of μ for which the following conditions are satisfied:

- a1) $\mu \in \tau_\mu$ and $\chi_\phi^m \in \tau_\mu$, where $\chi_\phi^m(x) = \prod_{j \in J} 0$;
- a2) $\lambda \cap \eta \in \tau_\mu$, whenever $\lambda, \eta \in \tau_\mu$;
- a3) if $\{\eta_a\}_{a \in \Lambda}$ is any collection of τ_μ , then $\bigcup_{a \in \Lambda} \eta_a \in \tau_\mu$, where Λ is an index set.

τ_μ And (M, n, τ_μ) are called the μ -topology on M and the relative topological space, respectively.

The observer a is called the constant observer on M if at least one of its elements is constant, i.e. for some $j \in J$, $a_j(x)$ is constant for all $x \in M$. The collection of all constant observers on M is denoted by C . Suppose that a and b be two constant observers on M for which $a < b$. If $\lambda \in \tau_\mu$, then $\lambda^{-1}(a, b)$ is defined as

$$\{x \in M \mid a_j(x) < \lambda_j(x) < b_j(x) \quad \forall j \in J\}.$$

Moreover the topology that is generated by $\{\lambda^{-1}(a, b) \mid \lambda \in \tau_\mu\}$ over M is denoted by $(\tau_\mu)_a^b$.

Let $S_1 \subseteq \bigcup_{a, b \in C} (\tau_\mu)_a^b$, $S_1 \neq \Phi$, and

$$S := \{U \mid U = U_0 \cap \mu^{-1}(a, b) \text{ \& } a, b \in C \text{ \& } U_0 \in S_1\}.$$

A pair (U_ϕ, ϕ) is called a chart for M if $U_\phi \in S$ and ϕ is a one to one mapping over U_ϕ so that $\phi(U_\phi)$ is a C^n Banach manifold. The set $D = \{(U_\phi, \phi) \mid (U_\phi, \phi) \text{ is a chart for } M\}$ is called a $C^n \mu$ -structure if

- b1) $\mu(M) = \mu(\bigcup_{(U_\phi, \phi) \in D} U_\phi)$;
- b2) if $(U_{\phi_1}, \phi_1) \in D$ for $i \in \{1, 2\}$ so that $\mu(U_{\phi_1}) = \mu(U_{\phi_2})$, then there exist a one to one and onto map $h_0: U_{\phi_1} \rightarrow U_{\phi_2}$ and a C^n Banach diffeomorphism $h: \phi_1(U_{\phi_1}) \rightarrow \phi_2(U_{\phi_2})$ such that $h \circ \phi_1 = \phi_2 \circ h_0$, where the restriction of h_0 to $U_{\phi_1} \cap U_{\phi_2}$ is the identity map.

Let E be a Banach space. (M, D) is called a μ -selective Banach manifold modeled over E if D is a $C^n \mu$ -structure.

1.1 Definition

Let (M, D_i) be $C^n \mu_i$ -selective Banach manifold for $i \in \{1, 2\}$. We write $D_1 \leq D_2$ if there exists an injective mapping $f: \bigcup_{(U_\phi, \phi) \in D_1} U_\phi \rightarrow \bigcup_{(V_\psi, \psi) \in D_2} V_\psi$ such that $\mu_1(U_\phi) = \mu_2(f(U_\phi))$ and $(f(U_\phi), \phi \circ f^{-1}) \in D_2$, where $(U_\phi, \phi) \in D_1$. D_1 and D_2 are called equivalent if $D_1 \leq D_2$ and $D_2 \leq D_1$, where D_i is $C^n \mu_i$ -structure, and $i \in \{1, 2\}$.

1.2 Definition

Let (M, D) be a $C^n \mu$ -selective Banach manifold. There exists a $C^n \mu$ -structure, A , on M such that every $C^n \mu$ -structure, D_0 , on M with $A \leq D_0$ is equivalent to A .

Any $C^n \mu$ -structure, A , that satisfies the conditions of Theorem 1.2 is called a C^n maximal μ -structure.

1.3 Definition

(M, A, μ) is called a C^n selective Banach manifold if A is a C^n maximal μ -structure on M and $\mu(M \setminus \bigcup_{(U_\phi, \phi) \in A} U_\phi) = \{0\}$. In this case, A is called a $C^n \mu$ -atlas.

1.4 Theorem

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold for $i \in \{1, 2\}$. Then, $(M_1 \times M_2, \Omega)$ is a $C^n \mu$ -selective Banach manifold, where

$$\Omega = \{(U_{\phi_a} \times V_{\psi_b}, \phi_a \times \psi_b) \mid (U_{\phi_a}, \phi_a) \in D_{M_1} \text{ \& } (V_{\psi_b}, \psi_b) \in D_{M_2}\};$$

whenever

$$\phi_a \times \psi_b: U_{\phi_a} \times V_{\psi_b} \rightarrow (\phi_a \times \psi_b)(U_{\phi_a} \times V_{\psi_b})$$

$$(\phi_a \times \psi_b)(x, y) = (\phi_a(x), \psi_b(y));$$

and

$$\mu: M_1 \times M_2 \rightarrow [0, 1] \times [0, 1]$$

$$\mu(x, y) = (\mu_1(x), \mu_2(y)).$$

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold that is modeled over Banach space E_i , and $a \in \text{Im } \mu_i$, where $i \in \{1, 2\}$. For every $a \in [0, 1]$, and every mapping $f: M_1 \rightarrow M_2$, put $K_{f, a} = \{(U_\phi, \phi, V_\psi, \psi) \mid (U_\phi, \phi) \in D_{M_1}, (V_\psi, \psi) \in D_{M_2}, f(U_\phi) \subseteq V_\psi, \mu_1(U_\phi) = \mu_2(V_\psi) = a\}$.

1.5 Definition

Let the mapping $f: M_1 \rightarrow M_2$ be as above, $p \in M$ and $r \leq 0$. We say that f is (r, a) -differentiable at p if the following conditions are satisfied:

- c1) $K_{f, a} \neq \Phi$;
- c2) $\mu_2 \circ f = \mu_1$;
- c3) if $(V_\psi, \psi) \in D_{M_2}$ and $\mu_2(V_\psi) = a$, then $\mu_2(V_\psi) = \mu_1(f^{-1}(V_\psi))$;

c4) if $(U_\phi, \phi; V_\psi, \psi) \in K_{f,a}$, then the mapping $\psi \circ f \circ \phi^{-1} : \phi(U_\phi) \rightarrow \psi(V_\psi)$ is a C^r -map in a neighborhood of $\phi(p)$.

Condition (c4) of Definition 1.5 implies that the definition of the (r, a) -differentiable mapping is independent of the choice of the charts. We say that the mapping f is continuous at p if $\psi \circ f \circ \phi^{-1}$ is a continuous map at $\phi(p)$. We say that the mapping f is (r, a) -differentiable on M if f is (r, a) -differentiable at every $p \in M$. If $r = \infty$, then we say that f is a -smooth.

1.6 Theorem

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold for $i \in \{1, 2\}$, and $f : M_1 \rightarrow M_2$ is a map. If $(V_\psi, \psi) \in D_{M_2}$, then there exists $(U_\phi, \phi) \in D_{M_1}$ such that $f^{-1}(V_\psi) = U_\phi$.

1.7 Theorem

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold over a Banach space E_i for $i \in \{1, 2, 3\}$, and $f : M_1 \rightarrow M_2$ and $g : M_2 \rightarrow M_3$ be two (r, a) -differentiable maps. Then, the mapping $g \circ f : M_1 \rightarrow M_3$ is (r, a) -differentiable.

2. Tangent Space for $C^n \mu$ -Selective Banach Manifolds

In this section, we discuss the concept of the a -level tangent space to a $C^n \mu$ -selective Banach manifold at a given point, and study its properties. Thorough out this section, we assume that every vector space is defined over the field F , where $F = \mathbb{R}$ or $F = \mathbb{C}$.

Let (M, A, μ) be a C^n selective Banach manifold over a Banach space E , $n \geq 1$, and the mapping

$$\mu : M \rightarrow \prod_{j \in J} I_j;$$

is an observer, where $I_j = [0, 1]$, and $J = \{1, \dots, m\}$. For every $p \in M$ and $a \in \text{Im } \mu$, we define an (r, a) -differentiable multi-path through p as a multi-function $\gamma : (-1, 1) \rightarrow M$ that satisfies the following conditions:

- d1) $\gamma(p) = 0$;
- d2) $\phi \circ \gamma : (-1, 1) \rightarrow E$ is a C^r map for every $(U_\phi, \phi) \in A$;

where $U_\phi \cap \gamma(-1, 1) \neq \emptyset$, $\mu(U_\phi) = a$, and $r \geq 1$. We denote the set of all (r, a) -differentiable multi-paths through p by $V_p^{r,a}$.

Let (M, D_M) be a $C^n \mu$ -selective Banach manifold over a Banach space E , and $p \in M$. Given two (r, a) -differentiable multi-paths γ and η passing through p , we write $\gamma \sim \eta$ if

$$(\phi \circ \gamma)'(0) = (\phi \circ \eta)'(0); \quad (1)$$

for every chart $(U_\phi, \phi) \in A$ around p , where $U_\phi \cap \gamma(-1, 1) \neq \emptyset$, $U_\phi \cap \eta(-1, 1) \neq \emptyset$, and $\mu(U_\phi) = a$. This relation is an equivalent relation.

By an a -level geometric tangent vector of M at p , we mean the equivalence class $[\gamma]$ of an (r, a) -differentiable multi-path γ passing through p . We define the a -level geometric tangent space of M at p as the set $\frac{V_p^{r,a}}{\sim}$ of all a -level geometric tangent vector of M at p , we denote it by $T_p^{\mu,a}(M)$.

Let γ be an (r, a) -differentiable multi-path passing through p , $(U_\phi, \phi) \in A$, $U_\phi \cap \gamma(-1, 1) \neq \emptyset$, and $\mu(U_\phi) = a$. We define the j th component of γ by

$$\gamma_j^{U_\phi} : (-1, 1) \rightarrow U_\phi \text{ with } \gamma_j^{U_\phi}(t) = \gamma(t) \cap U_\phi;$$

where $t \in \gamma^{-1}(U_\phi)$, $j \in J$, and $\text{Card} J < \infty$.

Let $r \geq 1$. The mapping $\gamma_j^{U_\phi}$ is a C^r map for every $j \in J$. Restriction $\sim$ to U_ϕ , we have $[\gamma_j^{U_\phi}] \in T_p(U_\phi)$. We define

$$A^{p,a} = \{(U_\phi, \phi) \in A \mid \mu(U_\phi) = a, p \in U_\phi\}.$$

Now, we define $\sim'$ as follows: $(U_{\phi_1}, \phi_1) \sim' (U_{\phi_2}, \phi_2)$ if the following conditions are satisfied:

- e1) $\phi_1(U_{\phi_1} \cap U_{\phi_2})$ is a Banach embedded submanifold of $\phi_1(U_{\phi_1})$;
- e2) there exists a Banach diffeomorphism $f_i : \phi_1(U_{\phi_1} \cap U_{\phi_2}) \rightarrow \phi_1(U_{\phi_1})$ for $i \in \{1, 2\}$.

It is straight forward to show that there exists a bijection $h : T_p^{\mu,a}(M) \rightarrow \prod_{[U_\phi, \phi]} T_p(U_\phi)$ that gives a vector space structure to $T_p^{\mu,a}(M)$ by transferring the structure of $\prod_{[U_\phi, \phi]} T_p(U_\phi)$ to $T_p^{\mu,a}(M)$.

2.1 Theorem

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold and $a \in \text{Im } \mu_i$ for $i \in \{1, 2\}$, and $f : M_1 \rightarrow M_2$ be an (r, a) -differentiable mapping at $p \in M_1$. We denote the a -level

differentiation of f at p by $d_p^a(f)$, and define it as the linear mapping

$$d_p^a(f): T_p^{\mu_1, a}(M_1) \rightarrow T_{f(p)}^{\mu_2, a}(M_2)$$

$$[\gamma] \mapsto [f \circ \gamma];$$

where $\gamma: (-1, 1) \rightarrow M_1$, and $\gamma(0) = p$.

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold modeled over a Banach space E_p and $a \in \text{Im } \mu$ for $i \in \{1, 2\}$. Then, $M_1 \times M_2$ is a $C^n \mu_i$ -selective Banach manifold that is modeled over the Banach space $E_1 \times E_2$ where

$$\mu: M_1 \times M_2 \rightarrow [0, 1] \text{ by } \mu(x, y) = (\mu_1(x), \mu_2(y)).$$

Let $(m_1, m_2) \in M_1 \times M_2$ be given. We consider multi-path γ passing through p by:

$$\gamma: (-1, 1) \rightarrow M_1 \times M_2$$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)): \quad \gamma(0) = (m_1, m_2);$$

where $\gamma_i: (-1, 1) \rightarrow M_i$, and $\gamma_i(0) = m_i$ for $i \in \{1, 2\}$.

2.2 Definition

A multi-path $\gamma: (-1, 1) \rightarrow M_1 \times M_2$ passing through $p = (m_1, m_2) \in M_1 \times M_2$ is an (r, a) -differentiable map if $(\phi \times \psi) \circ \gamma: (-1, 1) \rightarrow E_1 \times E_2$ is a C^r map for every chart $(U_\phi \times V_\psi, \phi \times \psi) \in D_{M_1 \times M_2}$, where $(U_\phi \times V_\psi) \cap \gamma((-1, 1)) \neq \Phi$, and $\mu(U_\phi \times V_\psi) = (a, a)$.

$(\phi \times \psi) \circ \gamma: (-1, 1) \rightarrow E_1 \times E_2$ is differentiable iff $(\phi \times \psi) \circ \gamma_i: (-1, 1) \rightarrow E_i$ is differentiable for $i \in \{1, 2\}$. Thus

$$\frac{d\gamma}{dt}(t) = \left(\frac{d\gamma_1}{dt}(t), \frac{d\gamma_2}{dt}(t) \right).$$

We denote the set of all (r, a) -differentiable multi-paths passing through $p = (m_1, m_2)$ by $W^{p, a}$.

We define the relation $\sim$ as follows:

$$\gamma \sim \eta \iff \frac{d(\beta \circ \gamma)}{dt}(0) = \frac{d(\beta \circ \eta)}{dt}(0);$$

where $(U_\beta, \beta) \in D_{M_1 \times M_2}$, $U_\beta \cap \gamma((-1, 1)) \neq \Phi$, $U_\beta \cap \eta((-1, 1)) \neq \Phi$, $\mu(U_\beta) = (a, a)$. It is easy to check that $\sim$ is an equivalence relation on $W^{p, a}$. We denote $\frac{W^{p, a}}{\sim}$ by $T_p^{\mu, a}(M_1 \times M_2)$, and we call it the a -level geometric tangent space to $M_1 \times M_2$ at $p = (m_1, m_2)$.

2.3 Theorem

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold over the Banach space E_p and $a \in \text{Im } \mu_i$ for $i \in \{1, 2\}$. Then, there exists a one to one correspondence between $T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2)$ and $T_{m_1}^{\mu_1, a}(M_1) \times T_{m_2}^{\mu_2, a}(M_2)$ that gives a vector space structure to $T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2)$ by transferring the structure of $T_{m_1}^{\mu_1, a}(M_1) \times T_{m_2}^{\mu_2, a}(M_2)$ to $T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2)$.

PROOF.

Let

$$f: T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2) \rightarrow T_{m_1}^{\mu_1, a}(M_1) \times T_{m_2}^{\mu_2, a}(M_2)$$

$$f([\gamma]) = ([\gamma_1], [\gamma_2]);$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, and $t \in (-1, 1)$. Clearly, f gives to $T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2)$ a vector space structure by transferring the structure of $T_{m_1}^{\mu_1, a}(M_1) \times T_{m_2}^{\mu_2, a}(M_2)$ to $T_{(m_1, m_2)}^{(\mu_1 \times \mu_2), a}(M_1 \times M_2)$.

2.4 Definition

Let (M_i, D_{M_i}) be a $C^n \mu_i$ -selective Banach manifold modeled over E_p , (N_i, D_{N_i}) be a $C^n \gamma_i$ -selective Banach manifold modeled over F_p , $a \in \text{Im } \mu_p$ and $a \in \text{Im } \gamma_i$ for $i \in \{1, 2\}$. We say that $f: M_1 \times M_2 \rightarrow N_1 \times N_2$ is (r, a) -differentiable if the following conditions are satisfied:

- f1) $(\gamma_1 \times \gamma_2) \circ f = \mu_1 \times \mu_2$;
- f2) there exists $(U_\phi, \phi) \in D_{M_i}$ and $(V_{\psi_i}, \psi_i) \in D_{N_i}$ so that $\mu_i(U_\phi) = \gamma_i(V_{\psi_i}) = a$ and $f(U_\phi \times U_{\phi_2}) \subseteq V_{\psi_1} \times V_{\psi_2}$, where $i \in \{1, 2\}$;
- f3) if $(W_\eta \times Z_\delta, \eta \times \delta) \in D_{N_1 \times N_2}$, and $a = \gamma_1(W_\eta) = \gamma_2(Z_\delta)$, then $\gamma_1 \times \gamma_2 = \mu_1 \times \mu_2(f^{-1}(W_\eta \times Z_\delta))$;
- f4) $(\eta \times \delta) \circ f \circ (\phi \times \psi)^{-1}: (\phi \times \psi)(U_\phi \times U_{\phi_2}) \rightarrow (\eta \times \delta)(V_{\psi_1} \times V_{\psi_2})$ is a C^r map between Banach spaces.

Let (M, A, μ) be a C^n selective Banach manifold. For given $p \in M$, we define the relation $\sim'$ on

$$A^{p, a} = \{(U_\phi, \phi) \in A \mid p \in U_\phi \text{ and } \mu(U_\phi) = a\};$$

as follows: $(U_{\phi_1}, \phi_1) \sim' (U_{\phi_2}, \phi_2)$ if

g1) $\phi(U_{\phi_1} \cap U_{\phi_2})$ is a Banach embedded submanifold of $\phi_i(U_{\phi_i})$;

g2) there exists a Banach diffeomorphism $f_i : \phi_i(U_{\phi_1} \cap U_{\phi_2}) \rightarrow \phi_i(U_{\phi_i})$;

for $i \in \{1, 2\}$.

Clearly, $\sim'$ is an equivalence relation on $A^{p,a}$. By the use of a choice function, we can choose an element of each class. We denote the set of these elements by $B^{p,a}$.

2.5 Definition

Let $f : M \rightarrow \bigcup_{p \in M} (\prod_{(U_{\phi}, \phi) \in B^{p,a}} R)$ be a mapping. We say that f is (r, a) - level differentiable if $f(p) \in \prod_{(U_{\phi}, \phi) \in B^{p,a}} R$ and $\pi_{\phi} \circ f \circ \phi^{-1}$ is a C^r map, where $(U_{\phi}, \phi) \in B^{p,a}$, and π_{ϕ} is the projection on the component (U_{ϕ}, ϕ) . We denote the set of such mappings by $G^{r,a}(M)$. $G^{r,a}(M)$ endowed with the multiplication and the addition of functions in components is a vector space over R .

3. Selective (r, a) - Vector Fields

In this section, we put forward then notion of selective (r, a) - vector fields. We suppose that (M, A, μ) is a C^{r+1} selective Banach manifold for $r \geq 1$.

3.1 Definition

A mapping $X : G^{r+1,a}(M) \rightarrow G^{r,a}(M)$ is called a selective (r, a) - vector field with observer μ if:

- h1) $X(fg) = X(f)g + fX(g)$ for $f, g \in G^{r+1,a}$;
- h2) $X(af + g) = aX(f) + X(g)$, where $a \in R$, and $f, g \in G^{r+1,a}(M)$.

3.2 Theorem

Let $\zeta^{r+1}(M)$ be the set of C^{r+1} functions on M , and X be an (r, a) - vector field on M . Define the mapping

$$X_p : \zeta^{r+1}(M) \rightarrow \prod_{(U_{\phi}, \phi) \in B^{p,a}} R;$$

by

$$X_p(f) = X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f)(p);$$

where $p \in M$. Then $X_p \in T_p^{\mu,a}(M)$.

PROOF.

Let $f, g \in \zeta^{r+1}(M)$ be given. Then

$$\begin{aligned} X_p(fg) &= X((\prod_{(U_{\phi}, \phi) \in B^{p,a}} f) \cdot (\prod_{(U_{\phi}, \phi) \in B^{p,a}} g))(p) \\ &= X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f)(p) \cdot (\prod_{(U_{\phi}, \phi) \in B^{p,a}} g(p)) \\ &\quad + (\prod_{(U_{\phi}, \phi) \in B^{p,a}} f(p)) \cdot X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} g)(p) \\ &= X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f)(p)g(p) + f(p)X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} g)(p). \end{aligned}$$

It is easy to check that X_p is linear. Therefore, $X_p \in T_p^{\mu,a}(M)$.

The next theorem gives a description of X over $U_{\phi,p} = \prod_{(U_{\phi}, \phi) \in B^{p,a}} U_{\phi}$.

3.3 Theorem

The mapping X is an (r, a) - vector field on $U_{\phi,p}$ if there exists an (r, a) - vector field $X^{U_{\phi}}$ defined on U_{ϕ} such that $X = \prod X^{U_{\phi}}$.

PROOF.

Suppose that the (r, a) - vector field $X^{U_{\phi}}$ be given, and $X = \prod_{(U_{\phi}, \phi) \in B^{p,a}} X^{U_{\phi}}$. Then for given $f, g \in G^{r+1}(U_{\phi,p})$, we have

$$\begin{aligned} X(fg) &= X((\prod_{(U_{\phi}, \phi) \in B^{p,a}} f^{U_{\phi}}) \cdot (\prod_{(U_{\phi}, \phi) \in B^{p,a}} g^{U_{\phi}})) \\ &= X(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f^{U_{\phi}} g^{U_{\phi}}) \\ &= (\prod_{(U_{\phi}, \phi) \in B^{p,a}} X^{U_{\phi}})(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f^{U_{\phi}} g^{U_{\phi}}) \\ &= \prod_{(U_{\phi}, \phi) \in B^{p,a}} X^{U_{\phi}}(f^{U_{\phi}} g^{U_{\phi}}) \\ &= \prod_{(U_{\phi}, \phi) \in B^{p,a}} (X^{U_{\phi}}(f^{U_{\phi}})g^{U_{\phi}} + f^{U_{\phi}}X^{U_{\phi}}(g^{U_{\phi}})) \\ &= ((\prod_{(U_{\phi}, \phi) \in B^{p,a}} X^{U_{\phi}})(\prod_{(U_{\phi}, \phi) \in B^{p,a}} f^{U_{\phi}})) \prod_{(U_{\phi}, \phi) \in B^{p,a}} g^{U_{\phi}} \\ &\quad + \prod_{(U_{\phi}, \phi) \in B^{p,a}} f^{U_{\phi}} ((\prod_{(U_{\phi}, \phi) \in B^{p,a}} X^{U_{\phi}})(\prod_{(U_{\phi}, \phi) \in B^{p,a}} g^{U_{\phi}})) \\ &= X(f)g + fX(g). \end{aligned}$$

The second property is straightforward from calculus. Therefore, X is an (r, a) - vector field.

Conversely, let X be an (r, α) - vector field on $U_{\phi, p}$ and $(U_{\phi}, \phi) \in B^{p, \alpha}$. We define the mapping $X^{U_{\phi}}$ on $U_{\phi, p}$ by

$$X^{U_{\phi}}(f) = \prod_{(U_{\phi}, \phi)} (X(\prod_{(U_{\phi}, \phi) \in B^{p, \alpha}} f)).$$

It is to check that $X^{U_{\phi}}$ is an (r, α) - vector field on $U_{\phi, p}$ and $X = \prod_{(U_{\phi}, \phi) \in B^{p, \alpha}} X^{U_{\phi}}$.

3.4 Definition

The α - level tensor product for $T_p^{\mu, \alpha}(M)$ and $(T_p^{\mu, \alpha}(M))^*$ is a vector space $T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^*$ endowed with a mapping

$$\tau: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^*$$

such that for every bilinear mapping

$$\varphi: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow X$$

where X is a vector space, there exists a linear mapping

$$\phi: T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^* \rightarrow X$$

So that the diagram of Figure 1 commutes: i.e. $\phi = \phi \circ \tau$.

3.5 Theorem

The α - level tensor products $T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^*$ are unique up to isomorphism approximation. That is for two α - level tensor products $\tau_1: T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^* \rightarrow T_1$ and $\tau_2: T_p^{\mu, \alpha}(M) \otimes (T_p^{\mu, \alpha}(M))^* \rightarrow T_2$ there exists a unique isomorphism $i: T_1 \rightarrow T_2$ such that the diagram of Figure 2 commutes: i.e. $\tau_2 = i \circ \tau_1$.

PROOF.

First, we show that for a given α - level tensor product $\tau: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow T$, the only mapping f :

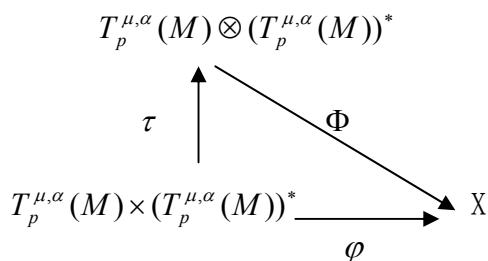


Figure 1. The diagram of the existence of $T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^*$

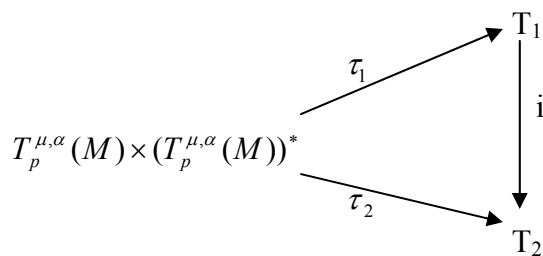


Figure 2. The diagram of the uniqueness of $T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^*$

$T \rightarrow T'$ which is compatible with τ is the identity mapping $id_T: T \rightarrow T$. That is the diagram of Figure 3 commutes, where $f = id_T$. In fact, for the given bilinear mapping $\tau: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow T$, there exists a linear mapping $\phi: T \rightarrow T$ such that the diagram of Figure 4 commutes. Since the identity mapping satisfies this condition, then by uniqueness of $\phi: T \rightarrow T$ we have $\phi = id_T$. Now, using Definition 3.4 for the case that $\varphi: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow X$ is replaced by $\tau_2: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow T_2$, results that the diagram of Figure 5 commutes for a unique linear mapping $\phi_1: T_1 \rightarrow T_2$. Similarly, for $\tau_1: T_p^{\mu, \alpha}(M) \times (T_p^{\mu, \alpha}(M))^* \rightarrow T_1$, there exists a unique linear mapping $\phi_2: T_2 \rightarrow T_1$ so that the diagram of Figure 6 commutes. Therefore, the

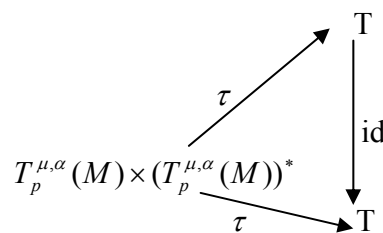


Figure 3. The diagram of $f = id_T$

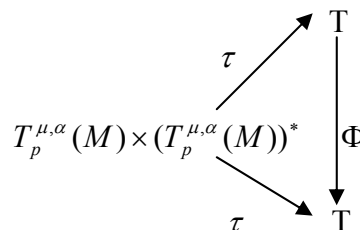


Figure 4. The diagram of $\phi = id_T$

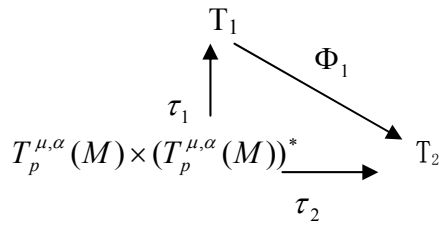


Figure 5. The diagram of the existence and uniqueness of $\varphi_1 : T_1 \rightarrow T_2$

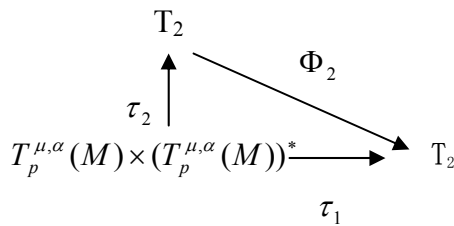


Figure 6. The diagram of the existence and uniqueness of $\varphi_2 : T_2 \rightarrow T_1$

mapping $\phi_2 \circ \phi_1 : T_1 \rightarrow T_1$ is compatible with the map τ_1 . Thus, $\phi_2 \circ \phi_1 : T_1 \rightarrow T_1$ is the identity mapping. Similarly, $\phi_1 \circ \phi_2 : T_2 \rightarrow T_2$ is

The identity mapping since it is compatible with the map τ_2 . Thus, $(\phi_1)^{-1} = \phi_2$ i.e. ϕ_1 and ϕ_2 are invertible. Therefore, both ϕ_1 and ϕ_2 are isomorphisms.

4. Conclusion

In this paper, the concept of tangent space for finite product of C^n μ -selective Banach manifolds is put forward. Next, the concept of differentiation of the mappings $f : M \rightarrow N$, where M and N are γ -selective and μ -selective Banach manifolds respectively, is extended

to the differentiation of the mappings $g : M_1 \times M_2 \rightarrow N_1 \times N_2$, where M_i and N_i are γ_i -selective and μ_i -selective Banach manifolds for $i \in \{1, 2\}$, respectively. Moreover, the notions of vector field and tensor field over μ -selective Banach manifolds are introduced.

5. References

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