ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645
Indian Journal of Science and Technology, *Vol 8(6), 518–528, March 2015*
DOI : 10.17485/ijst/2015/v8i6/61064

Single Variables Control Charts: A Further Overview

Keoagile Thaga* **and Ramasamy Sivasamy**

Department of Statistics, University of Botswana, Private Bag 0022, Gaborone, Botswana; thagak@mopipi.ub.bw

Abstract

Control charts are widely used in industries to monitor a process for quality improvement. When dealing with variables data, we usually employ two control charts to monitor the process location and spread since one of them or both may change. Furthermore, a change in the process variability can affect effectiveness of the control chart for the mean as its control limits will change. We give an overview of control charts that are proposed in the last decade or so in an effort to use only one chart to simultaneously monitor both process location and spread. Two approaches have been advocated for using one control chart for process monitoring. One approach plots two quality characteristics in the same chart while the other uses one plotting variable to represent the process location and spread.

Keywords: Autocorrelated Processes, Single Chart, Statistical Process Control, Variables Data

1. Introduction

This paper is an extension of an overview paper by Cheng and Thaga[3] in which they presented some single variable control charts. More charts have been developed thereafter and we present a further overview of the charts in the literature. Control charts are very important tools in statistical process control whose main objective is to improve the quality of processes so as to satisfy customer requirements. One of the main concerns about the implementation of control charts for process monitoring is the difficulty in plotting and interpreting them. This is due to the fact that process monitoring is usually carried out by shop-floor or line workers. These line workers usually have inadequate training in using control charts and interpreting control charts signals. Several charting techniques have been proposed to try to simplify the process of statistical quality control, in particular the Shewhart control chart developed by Shewhart[16], Cumulative Sum (CUSUM) control chart by Page[14] and the Exponentially Weighted Moving Average (EWMA) control chart developed by Roberts[15]. The Shewhart chart is effective for detecting large process shifts while the CUSUM and EWMA charts are effective for detecting small and persistent shifts in the process mean and/or standard deviation.

Most of the Shewhart, CUSUM, and EWMA charts for variables data discussed in the literature monitor the process location and spread separately. Two control charts, one for monitoring the process center (such as the $\overline{X}$ chart) and the other for monitoring the process spread (such as an R chart or S chart), are run concurrently. This practice requires more resources such as quality control practitioners, time and other resources needed for process monitoring. Recently, some effort has been committed to designing a control chart that can simultaneously monitor both process mean and variability. Such a chart is called a single variables control chart. Some of the main difficulties encountered in this endeavor include designing a single chart that is effective for both small and large shifts in both parameters, designing a chart that is simple to use and interpret, and designing a chart that when an out-of-control signal is issued, it can immediately indicate whether the mean is out of control, the variability is out of control or both, as well as the direction of the shift.

2. Simultaneous Control Charts

Simultaneous charts are constructed by plotting two quality characteristics in the same chart, one for the mean and another for the standard deviation. White and Schroeder[22] first introduced the use of one control chart

*Author for correspondence

to monitor both process mean and variability. This chart was developed using resistant measures and a modified box plot display. The control limits for this chart are determined by the median and a resistant measure of spread called the Q-spread. This chart is difficult to interpret, particularly the Q-spread charts. Domangue and Patch[11] developed some omnibus exponentially weighted moving average schemes based on the exponentiation of the absolute value of the standardized sample mean of observations, capable of simultaneously detecting shifts in the process mean and process standard deviation. These charts are sensitive to shifts in the process mean and/or variability. However, it is not possible to identify whether the change has actually occurred in the process mean or in the process variability. For multivariate processes, Spiring and Cheng[17] developed a single chart that plots both process mean and standard deviation on the same chart. This chart also plots two variables in the same chart.

Lu and Reynolds[13] proposed a simultaneous EWMA chart for autocorrelated processes. They proposed a chart that is developed by fitting a time series model to the process observations and then computing the residuals. These residuals are then plotted on standard control charts, because the residuals are independent and identically distributed normal random variables when the process is in control, when the fitted time series model is the same as the true process model and the parameters are estimated without error. This chart requires using either two charts concurrently or plotting two variables in the same chart with two control limits. One control limit is for the process mean while the other is for the process variability. An out-of-control signal is issued if either one of the plotted variables plots outside its acceptance region.

3. Single Control Charts

In recent years, Cheng in collaboration with other authors concentrated their effort to developing control charts that use only one plotting character for monitoring both process mean and standard deviation on a single chart. Such charts are referred to as single variables control charts. These charts are developed by first transforming the plotting statistics for the process mean and standard deviation to standard normal random variables. The maximum of these transformed variables is then plotted against time or sample number on a control chart. When there is a change in the process, the single chart should be able to issue an out-of-control signal and identify the parameter (s) that has shifted as well as the direction of the shift.

Cheng and Li[9], proposed a single variables T control chart. This chart measures the proximity of the observations to the target value (center) and the variability of the process. The T chart suffers from the weakness of not being able to tell which parameter has shifted when an out-of-control signal is issued. Chao and Cheng[1] developed a single control chart called the semicircle control chart. This chart uses a semicircle to plot a single plotting character to indicate the position of the mean and standard deviation by plotting the two parameters against each other. When an out-of-control signal is issued, the chart is capable of showing the parameter (s) that has shifted from its target value. The disadvantage of this chart is that it loses track of the time sequence of the plotted points.

Chen and Cheng[8] developed a single Shewhart-type control chart called the Max chart. This chart plots the maximum absolute values of the standardized mean and standard deviation. It is capable of simultaneously monitoring the process mean and variability using a single plotting variable. When an out-of-control signal is issued, this chart shows the parameter (s) that has shifted as well as the direction of the shift. This chart performs like the combined Shewhart charts for the mean and standard deviation, i.e. the combined $\overline{X}$ and S charts. The Max-chart is effective in detecting large shifts in both process mean and/or standard deviation. However, the Max chart is not very effective in detecting small shifts in the process mean and/or standard deviation.

Chen, Cheng and Xie[7] proposed the Max-EWMA chart which is capable of detecting small changes in the process. Xie[23] proposed the SS-EWMA chart, which is based on the sum of squares of the maximum standard EWMA values. These charts are easy to construct and it is easy to identify the time at which the shift occurred as well as the source and direction of the shift when an out-of-control signal is issued.

Xie[23] further proposed the EWMA-Max chart which is an extension of the Max chart proposed by Chen and Cheng[8]. This chart is capable of detecting small changes in the process mean and/or variability. Chen, Cheng and Xie[6] proposed the EWMA-SC chart by applying the EWMA techniques to the statistic employed in the semicircle chart. These charts are very sensitive to small changes in the process mean and/or standard deviation.

Xie[23] also proposed a single multivariate EWMA chart that can simultaneously monitor the process location and spread in the multivariate case as well as identify the source of an out-of-control signal. This chart is more sensitive to small changes in the process than the combined χ^2 and $|S|$ charts.

Cheng and Thaga[2] proposed the Max-CUSUM chart and the SS-CUSUM chart Thaga[18] that are capable of detecting small changes in the process mean and/or standard deviation for processes whose quality characteristics follow a univariate independent normal distribution.

Cheng and Thaga[4,5] also proposed the Multivariate Max-CUSUM chart, Multivariate Max chart, Thaga and Gabaitiri[21] and Max-EWMA Chart for Autocorrelated Processes, Thaga and Yadavalli[20] which can be used to monitor the process that is simultaneously characterized by two or more related quality characteristics. The Multivariate Max-CUSUM chart is very effective in monitoring small changes in the process mean vector and/or variability and the Multivariate Max chart quickly detects large changes in the process mean vector and/or variability while the Max-EWMA Chart for Autocorrelated Processes chart can quickly detects moderate to large shifts in the mean and/or standard deviation at both low and high levels of autocorrelations than the Max-CUSUM chart for autocorrelated processes.

In addition to the ability to detect these process changes, these charts are easy to implement and interpret as well as identify the source of an out-of-control signal when the process has changed.

Zhang, Li and Wang[24] proposed a New Adaptive Control Chart for Monitoring Process Mean and Variability. This chart integrates the EWMA procedure with the generalized likelihood ratio test statistics to jointly monitor the process mean and/or variability. This chart is effective in detecting the disturbances that shift the process mean increase or decrease in the process variability or a combination of both. Costa, A. B. F. and Rahim, M. A[10] proposed a Single EWMA Chart for Monitoring Process Mean and Process Variance which performs better than the combination of the EWMA $\overline{X}$ chart and the EWMA $\ln(S^2)$ chart except when there is a small shift in the mean while the process variability is in control.

Gan[12] proposed a simultaneous EWMA chart that is developed by combining a chart for the mean and a chart for the variance into one chart by plotting the EWMA of log (S^2) against the EWMA of $\overline{X}$ The control limits of this chart are formed by either a rectangle or an ellipse. When an out-of-control signal is issued, the magnitude, direction as well as the source of the process shift is indicated by the position of the point. However, the time sequence of the plotted points is lost. Plotting each point on a new chart when an out-of-control signal is issued can solve this problem. This is however more time consuming, especially for large processes.

The control charts discussed above are designed under the assumption that a process being monitored will produce measurements that are independent and identically distributed over time when only the inherent sources of variability are present in the process. However, in some applications, the assumption of independent observations is not realistic. For instance, measured variables from tanks, reactors and recycle streams in chemical processes show significant serial correlation. In some instances, the dynamics of the process will induce correlations in observations that are closely spaced in time. If the sampling interval used for process monitoring in these applications is short enough for the process dynamics to produce significant correlation, then this correlation can have very serious effects on the properties of standard control charts developed under the independence assumption.

Thaga, Gabaitiri and Kgosi[19] proposed single CUSUM and Shewhart-type charts for autocorrelated processes. These charts are developed by, first fitting a time series model to the process observations and then computing residuals using the fitted model. These residuals are then plotted on standard control charts, because the residuals are independent and identically distributed normal random variables when the process is in control, the fitted time series model is the same as the true process model and the parameters are estimated without error.

These proposed charts monitor the process by simultaneously monitoring the residual means and variation because a shift in the process mean and/or standard deviation causes a shift in the mean and/or standard deviation of the residuals. The results show that by adjusting the reference value of the standard CUSUM chart to take the autocorrelation structure

I. Shewhart-type Single Charts

Chart	Simultaneous chart (1987)	Single variables T chart (1993)	Semicircle chart (1996)	Alternate variables chart (1998)	Max chart (1998)
Statistics	Mean of medians = μ_m Variance of medians = σ_m^2 Mean of Q-spread = μ_Q Variance of Q-spread = σ_Q^2 Media: $\mu_m \pm K\sigma_m$ Q-Spread: $\mu_Q \pm K\sigma_Q$	Let T* be the target value. $Y = \frac{X-\mu}{\sigma}$, and $T = \frac{T^* - \mu}{\sigma}$. $Y_{(1)}, Y_{(2)}, \bullet\bullet\bullet, Y_{(n)}$ are the order statistics. Define the plotting statistic as $W = \lvert Y_{(1)} - T \rvert + \lvert Y_{(n)} - T \rvert$	Let S^* denote the Root mean square $T = (\bar{X} - \mu)^2 + S^{*2}$ $\frac{nT}{\sigma^2} \sim \chi_n^2$. T defines a circle, we plot $(\bar{X}, S_i^*)$ in a circle with radius $r = \left(\frac{\chi_n^2, (1-\alpha)}{n}\right)^{1/2} \sigma.$	T denotes the target value. $T^* = (\bar{x} - T)^2$ $MSE =$ $(n-1)^{-1} \sum_{i=1}^{n} (x_i - T)^2$ $S^2 = (n-1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ $UCL_{T^*} = n^{-1}\sigma^2 \chi^2_{1,\lambda,(1-\alpha)}$ $UCL_{MSE} = (n-1)^{-1}\sigma^2 \chi^2_{n,\lambda,(1-\alpha)}$ where $\lambda = n[(\mu - T)/\sigma]^2$ $UCL_{S^2} = (n-1)^{-1}\sigma^2 \chi^2_{n-1,(1-\alpha)}$	$U_i = \frac{(\bar{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$ $V_i = \Phi^{-1}\left\{H\left(\frac{(n_i-1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$ $M_{(n_i)} = \max\{\lvert U_i \rvert, \lvert V_i \rvert\}$ a is a change in the process mean and b is the change in the process standard deviation. $P(M_{(n_i)} \le y) =$ $\left\{\begin{matrix}\left[\Phi\left(\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\right) - \right. \\ \left. \Phi\left(-\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\right)\right]\end{matrix}\right\}$ $\times \left\{\begin{matrix}\left[H\left(\frac{\chi^2_{\Phi(y),n_i-1}}{b^2}; n_i - 1\right)\right. \\ \left. -H\left(\frac{\chi^2_{\Phi(-y),n_i-1}}{b^2}; n_i - 1\right)\right]\end{matrix}\right\}$
Comments	This chart uses resistant measures and is robust against distributional assumptions. It requires two control limits. It cannot be used for samples of size one.	This chart measures the proximity to the target value and the variability of the process. It uses one plotting variable for both parameters. However, it does not show the source of an out-of-control signal.	It is easy to detect out-of-control signals and attribute them to the cause of the mean and/or variation. However, the chart loses track of the time sequence of the plotted points.	This chart provides the information about the process's proximity to the target values and its variability. It plots two variables in the same chart. It is effective for detecting large process changes.	This chart plots one variable in the chart. The source as well as direction of the shift is indicated when an out-of-control signal is issued. The chart is effective in detecting large shifts in the process.

II. EWMA-type Single Charts

Chart	Omnibus EWMA chart (1991)	Max-EWMA chart (2001)	SS-EWMA chart (1999)
Statistics	$A_i = r\|Z_i\|^{\alpha} + (1-r)A_{i-1}$ $0 < r \leq 1$ $E*(A_i) = \left(\frac{2^{\alpha}}{\pi}\right)^{1/2} \Gamma[(\alpha+1)/2]$ $\text{Var}*(A_i) = \frac{2^{\alpha} r}{(2-r)\pi}[\sqrt{\pi}\Gamma[\alpha+.5] - (\Gamma[(\alpha+1)/2])^2]$ $UCL = E^*(A_i) + LVar^*(A_i)$	$Z_i = \frac{(\bar{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$ $Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i-1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$ $U_i = (1-\lambda)U_{i-1} + \lambda Z_i$ $V_i = (1-\lambda)V_{i-1} + \lambda Y_i$ $M_i = \max[\|U_i\|, \|V_i\|]$ $F(y; \sigma_{U_i}) = P(M_i \leq y)$ $= \left[2\Phi(\frac{y}{\sigma_{U_i}})\right]^2, y \geq 0$ $UCL = E(M_i) + L\sqrt{Var(M_i)}$	$Z_i = \frac{(\bar{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$ $Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i-1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$ $U_i = (1-\lambda)U_{i-1} + \lambda Z_i$ $V_i = (1-\lambda)V_{i-1} + \lambda Y_i$ $SS_i = U_i^2 + V_i^2$ $\frac{U_i^2}{\sigma_{U_i}^2} + \frac{V_i^2}{\sigma_{V_i}^2} \sim \chi_2^2$
Comments	When an out-of-control signal is issued, one cannot immediately identify the source as well as the direction of the shift. The chart is sensitive to small process shifts.	This chart uses one plotting variable to monitor the process mean and/or variability. It is effective for detecting small shifts in the process. The source and direction of the shift can be easily identified.	We plot a pair of (U_i, V_i) on a chart defined by a circle with radius = $\sqrt{UCL}$. It is easy to detect out-of-control signals and attribute them to the cause of the mean and/or variation. However, the chart loses track of the time sequence of the plotted points.

into consideration, the CUSUM chart can effectively detect small shifts in the process mean and/or spread. The Shewhart-type single chart for autocorrelated data is effective in detecting large shifts in the process mean and/or standard deviation.

4. Summary of Statistics used in Single Variables Chart

Let $X_i = X_{i1}, X_{i2}, \bullet\bullet\bullet, X_{in_i}, i = 1, 2, \bullet\bullet\bullet,$ denote a sequence of samples of size n_i taken on a quality characteristic X. It is assumed that, for each *i*, $X_{i1}, X_{i2}, \bullet\bullet\bullet, X_{in_i}$, are independent and identically distributed observations following a normal distribution, where *i* indicate the i^{th} group. Let μ_0 and σ_0 be the nominal process mean and standard deviation previously established. Assume that the process parameters μ and σ can be expressed as $\mu = \mu_0 + a\sigma_0$ and $\sigma = b\sigma_0$ for $b > 0$, where $a \neq 0$ and $b \neq 1$. When the process is in control, $a = 0$ and $b = 1$, otherwise, the process has changed due to some assignable causes. The constants *a* and *b* represent shifts in the mean and standard deviation respectively.

EWMA-type Single Charts Continued

EWMA-Max chart (1999)	EWMA-SC chart (2004)	Joint EWMA (2000)
$Z_i = \frac{(\bar{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$ $Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i - 1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$ $G_i = \max\{\lvert Z_i \rvert, \lvert Y_i \rvert\}$ $W_i = (1-\lambda)W_{i-1} + \lambda G_i$ $\text{UCL} = \text{E}(W_i) + L\sqrt{Var(Y_i)}$ $= \text{E}(G_i) + L\sqrt{\frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}Var(G_i)}$	$T_i = (\bar{X}_i - \mu_0)^2 + \frac{n-1}{n}S_i^2$ $T_i^* = \frac{n}{\sigma_0^2}T_i \sim \chi_n^2$ $Q_i = (1-\lambda)Q_{i-1} + \lambda T_i^*$ $\text{UCL} = n + L\sqrt{\frac{2n\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}}$ OR $U_i = (1-\lambda)U_{i-1} + \lambda\left[\frac{n(\bar{X}_i - \mu_0)^2}{\sigma_0^2} - 1\right]$ $V_i = (1-\lambda)V_{i-1} + \lambda\left[(n-1)(\frac{S_i^2}{\sigma_0^2} - 1)\right]$ $\text{UCL} = L\sqrt{\frac{2n\lambda[1-(1-\lambda)^{2i}]}{2-\lambda}}$	$Q_t = (1-\lambda_m)Q_{t-1} + \lambda_m \bar{X}_t$ $q_t = (1-\lambda_v)q_{t-1} + \lambda_v \log(S_t^2)$ For the rectangular acceptance region chart, the process is out of control if $Q_t > h_m$ or $Q_t < -h_m$ and/or $q_t > h_v$ or $q_t < -h_v$ For the elliptical acceptance region (bull's-eye) chart, the process is out of control if $\frac{(Q_t - \mu_0)}{h_m^2} + \frac{\{q_t - E[(\log(S^2)]\}^2}{(h_v - E[\log(S^2)]\}^2} > 1$
This chart is an extension of the max chart. It indicates the source as well as direction of the shift when an out-of-control signal is issued. This chart is effective in detecting small shifts in the process mean and/or standard deviation.	This chart is an extension of the semicircle chart. The chart is constructed by either plotting the Q_i values against time or plotting a pair of (U_i, V_i) on a two dimensional chart. The chart is effective in detecting small changes in the process and indicating the source and direction of the changes when a signal is issued.	The interpretation of an out-of-control signal is easy because the position of the point gives an indication of both the magnitude and direction of the process shift. This chart suffers from the problem of not being able to show the time or sample at which the signal was issued.

EWMA-type Single Charts Continued

A Single EWMA Chart (2006)	A Single Adaptive EWMA chart (2012)
$\xi_i = \begin{cases} \text{d}, & \text{if } \text{e}_\text{i} \geq 0 \\ -d, & \text{if } \text{e}_\text{i} < 0 \end{cases}$ $T_i = \sum_{j=1}^{n}\left(X_{ij} - \mu_0 + \xi_i \sigma_0\right)^2, \quad i = 1, 2, \ldots,$ $W_i = (1-r)W_{i-1} + rT_i, \; i = 1, 2, \ldots,$ $UCL = k\sigma_0^2$	$l_t = n\left(\bar{x}_t^2 + S_t^2 - \ln S_{t-1}^2\right)$ $LR_t = \bar{x}_t^2 + S_t^2 - \ln(S_i^2)$ $u_t = \lambda \bar{x}_t + (1-\lambda)u_{t-1,}$ $v_t = \lambda S_t^{*2} + (1-\lambda)v_{t-1}$ $S_t^{*2} = \sum_{j=1}^{n}\left(x_{tj} - u_t\right)^2 / n$

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A Single EWMA Chart (2006)	A Single Adaptive EWMA chart (2012)
	$\mathrm{ELR}_t = u_t^2 + v_t - \ln(v_t), t = 1, 2, ...,$ Control Limits $d_i \begin{cases} d_1, & \text{if } 0 \le \mathrm{ELR}_{t-1} < g \\ d_2, & \text{if } g \le \mathrm{ELR}_{t-1} < h \end{cases}$
This chart is an extension of the new EWMA control chart for monitoring both location and dispersion. It performs better than the combination of the EWMA $\overline{X}$ chart and the EWMA $\ln(S^2)$ chart except when there is a small shift in the mean while the process variability is in control.	This chart is an extension of the adaptive EWMA chart. It is more effective in detecting the disturbances that shifts the process mean and/or variability

III. CUSUM type Single Charts

Chart	Max-CUSUM chart (2003)	SS-CUSUM chart (2003)
Statistics	$Z_i = \frac{(\overline{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$	$Z_i = \frac{(\overline{X}_i - \mu_0)}{\sigma_0 / \sqrt{n_i}}$
	$Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i-1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$	$Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i-1)S_i^2}{\sigma_0^2}; n_i - 1\right)\right\}$
	$C_i^+ = \max[0, Z_i - k + C_{i-1}^+],$	$C_i^+ = \max[0, Z_i - k + C_{i-1}^+],$
	$C_i^- = \max[0, -k - Z_i + C_{i-1}^-],$	$C_i^- = \max[0, -k - Z_i + C_{i-1}^-],$
	$S_i^+ = \max[0, Y_i - k + S_{i-1}^+],$	$S_i^+ = \max[0, Y_i - k + S_{i-1}^+],$
	$S_i^- = \max[0, -k - Y_i + S_{i-1}^-],$	$S_i^- = \max[0, -k - Y_i + S_{i-1}^-],$
	$M_i = \max[C_i^+, C_i^-, S_i^+, S_i^-]$	$M_i = \max[C_i^+, C_i^-],$
		$V_i = \max[S_i^+, S_i^-]$
		$SS_i = M_i^2 + V_i^2$
Comments	This chart plots one variable (M_i) in the chart with the decision interval h and reference value k, where k is half the shift we want to detect and h is determined by a combination of k and an in-control average run length. It indicates the source as well as direction of the shift when an out-of-control signal is issued. This chart is effective in detecting small shifts in the process mean and/or standard deviation.	This chart plots successive pairs of (V_i, M_i) values in a chart with control limits defined by a circle with radius equal to the decision interval. This chart is effective in detecting small shifts in the mean and/or standard deviation.

IV. Multivariate Single Charts

Chart	Multivariate Max-EWMA chart (1999)	Multivariate Max-CUSUM chart (2003)	Multivariate Max chart (2003)	Alternate variables chart (1998)
Statistics	$\mathbf{Z}_i = (1-\lambda)\mathbf{Z}_{i-1} + \lambda(\bar{\mathbf{X}}_i - \mu_0)$ $T_i = \frac{n(2-\lambda)}{\lambda[(1-\lambda)^{2i}]}\mathbf{Z}_i'\Sigma^{-1}\mathbf{Z}_i \sim \chi^2_{k,\delta^2}$ where, $\delta^2 = n(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)$ $U_i = \Phi^{-1}\left[H_k\left\{\frac{n(2-\lambda)}{\lambda[(1-\lambda)^{2i}]}\mathbf{Z}_i'\Sigma_0^{-1}\mathbf{Z}_i\right\}\right]$ $W_i = \sum_{j=1}^{n}(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)'\Sigma_0^{-1}(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)$ $Y_i = (1-\lambda)Y_{i-1} + \lambda\Phi^{-1}\{H_{k(n-1)}W_i\}$ $V_i = \sqrt{\frac{2-\lambda}{\lambda[(1-\lambda)^{2i}]}}Y_i$ $M_i = \max\{\lvert U_i\rvert, \lvert V_i\rvert\}$ $UCL = E(M_i) + L\sqrt{Var(M_i)}$	The statistics for changes in the mean vector from μ_G to μ_B are given as $a' = \frac{(\mu_B-\mu_G)'\Sigma^{-1}}{[(\mu_B-\mu_G)'\Sigma^{-1}(\mu_B-\mu_G)]^{1/2}}$ $D = \sqrt{(\mu_B-\mu_G)'\Sigma^{-1}(\mu_B-\mu_G)}$ $Z_i = a'(\mathbf{X}_i - \mu_G)$ $C_i^+ = \max[0, Z_i - 0.5D + C_{i-1}^+]$ $C_i^- = \max[0, -0.5D - Z_i + C_{i-1}^-]$ The statistics for changes in the covariance matrix from Σ to $b\Sigma$ are given as $Y_i = \Phi^{-1}\{(\mathbf{X}_i-\mu)'\Sigma^{-1}(\mathbf{X}_i-\mu); p\}$ $k = \log(b)\left(\frac{b}{b-1}\right)$ $S_i^+ = \max[0, Y_i - k + S_{i-1}^+]$ $S_i^- = \max[0, -k - Y_i + S_{i-1}^-]$ $M_i = \max[C_i^+, C_i^-, S_i^+, S_i^-]$	$T_n = n(\bar{\mathbf{X}}_n - \mu_0)'\Sigma_0^{-1}(\bar{\mathbf{X}}_n - \mu_0)$ $\lvert\mathbf{S}_n\rvert = s_1^2 s_2^2 - s_{12}^2$ $\delta = \frac{n}{b^2}(\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)$ $Z_i = \Phi^{-1}\{H[n(\bar{\mathbf{X}}_n-\mu_0)'\Sigma_0^{-1}(\bar{\mathbf{X}}_n-\mu_0); p]\}$ $Y_i = \Phi^{-1}\left\{H\left[\frac{2(n-1)\lvert S_n\rvert^{1/2}}{\lvert\Sigma_0\rvert^{1/2}}; 2n-4\right]\right\}$ $M_i = \max\{\lvert Z_i\rvert, \lvert Y_i\rvert\}$	$\mathbf{T}_p = (\bar{\mathbf{X}}-\mathbf{T})'\Sigma^{-1}(\bar{\mathbf{X}}-\mathbf{T})$ $\text{MSE}_p = (n-1)^{-1}\sum_{i=1}^{n}(\mathbf{X}_i-\mathbf{T})'\Sigma^{-1}(\mathbf{X}_i-\mathbf{T})$ $S_p^2 = (n-1)^{-1}\sum_{i=1}^{n}(\mathbf{X}_i-\bar{\mathbf{X}})'\Sigma^{-1}(\mathbf{X}_i-\bar{\mathbf{X}})$ $UCL_{\mathbf{T}_p} = n^{-1}\chi^2_{p,\lambda,(1-a)}$ $UCL_{MSE_p} = (n-1)^{-1}\chi^2_{np,\lambda,(1-a)}$ $UCL_{S_p^2} = (n-1)^{-1}\chi^2_{(n-1)p,(1-\alpha)}$
Comments	This chart is sensitive small changes in the mean vector and/or covariance matrix. It indicates the source of an out-of-control signal.	This chart is effective for monitoring small changes in the process mean vector and/ or covariance matrix than the multivariate Max-EWMA chart. It also shows the source of an out-of-control signal.	This chart is effective in detecting large shifts in the process mean vector and/or covariance matrix. It shows the parameter that has shifted when a signal is given.	This chart provides the information about the process's proximity to the target values and its variability. Unlike other charts discussed in this table, it requires plotting two variables in the same chart.

V. Single Charts for Autocorrelated Processes

Chart	Max-CUSUM chart (2003)	Shewhart-type (2003)	Combined EWMA chart (1999)
Statistics	The residual at time *t* is $e_t = X_t - \xi_0 + \varphi(X_{t-1} - \xi_0) + \theta e_{t-1}$. The asymptotic mean and variance of the residuals after the shift are respectively $\mathrm{E}(e_t) = \frac{1-\varphi}{1-\theta}(\xi_1 - \xi_0)$ $\mathrm{Var}(e_t) =$ $\sigma^2_{\gamma_0} + \frac{\varphi^2 - 2\varphi\theta + 1}{1-\theta_0^2}(\sigma^2_{\varepsilon 1} - \sigma^2_{\varepsilon 0}) + \frac{\sigma^2_{\alpha 1} - \sigma^2_{\alpha 0}}{1-\theta_0^2}$ $\bar{\xi}_i = (\xi_{i1} + \xi_{i2} + \bullet\bullet\bullet + \xi_{in_i}) / n_i$ $MSE_i = \sum_{j=1}^{n_i} (\xi_{ij} - \bar{\xi}_i)^2 / n_i$ $Z_i = \frac{(\bar{\xi}_i - \xi_0)}{\sigma_{\gamma_0} / \sqrt{n_i}}$ $Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i)MSE_i}{\sigma^2_{\gamma_0}}; n_i\right)\right\}$ $k = \left(\frac{1-\varphi}{1-\theta}\right)\delta / 2$ $C_i^+ = \max[0, Z_i - k + C^+_{i-1}]$, $C_i^- = \max[0, -k - Z_i + C^-_{i-1}]$, $S_i^+ = \max[0, Y_i - k + S^+_{i-1}]$, $S_i^- = \max[0, -k - Y_i + S^-_{i-1}]$, $M_i = \max[C_i^+, C_i^-, S_i^+, S_i^-]$	The residual at time t is $e_t = X_t - \xi_0 + \varphi(X_{t-1} - \xi_0) + \theta e_{t-1}$ The asymptotic mean and variance of the residuals after the shift are respectively $\mathrm{E}(e_t) = \frac{1-\varphi}{1-\theta}(\xi_1 - \xi_0)$ $\mathrm{Var}(e_t) =$ $\sigma^2_{\gamma_0} + \frac{\varphi^2 - 2\varphi\theta + 1}{1-\theta_0^2}(\sigma^2_{\varepsilon 1} - \sigma^2_{\varepsilon 0}) + \frac{\sigma^2_{\alpha 1} - \sigma^2_{\alpha 0}}{1-\theta_0^2}$ $\bar{\xi}_i = (\xi_{i1} + \xi_{i2} + \bullet\bullet\bullet + \xi_{in_i}) / n_i$ $MSE_i = \sum_{j=1}^{n_i} (\xi_{ij} - \bar{\xi}_i)^2 / n_i$ $Z_i = \frac{(\bar{\xi}_i - \xi_0)}{\sigma_{\gamma_0} / \sqrt{n_i}}$ $Y_i = \Phi^{-1}\left\{H\left(\frac{(n_i)MSE_i}{\sigma^2_{\gamma_0}}; n_i\right)\right\}$ $M_i = \max\{\lvert Z_i \rvert, \lvert Y_i \rvert\}$ $F(x_i; n_i; 0,1) = P(M_i \le x) = P(\lvert Z_i \rvert \le x)\, P(\lvert Y_i \rvert \le x)$ $= \{\Phi(x) - \Phi(-x)\}^2 = P(\chi_1^2 \le x^2)^2$	The residual at time t is $e_t = X_t - \xi_0 + \varphi(X_{t-1} - \xi_0) + \theta e_{t-1}$ The asymptotic mean and variance of the residuals after the shift are respectively $\mathrm{E}(e_t) = \frac{1-\varphi}{1-\theta}(\xi_1 - \xi_0)$ $\mathrm{Var}(e_t) =$ $\sigma^2_{\gamma_0} + \frac{\varphi^2 - 2\varphi\theta + 1}{1-\theta_0^2}(\sigma^2_{\varepsilon 1} - \sigma^2_{\varepsilon 0}) + \frac{\sigma^2_{\alpha 1} - \sigma^2_{\alpha 0}}{1-\theta_0^2}$ Control chart for the process mean is based on the following statistics $Y_t = (1-\lambda)Y_{t-1} + \lambda e_t$ the control limits are $\pm c\sqrt{\frac{\lambda}{(2-\lambda)}}\sigma_\gamma$ Control chart for the spread is based on $Y_t = \max[(1-\lambda)Y_{t-1} + \lambda \ln(e_t^2), \ln(\sigma^2_{\gamma_0})]$ with control limit given as $\ln(\sigma^2_{\gamma_0}) + c\sqrt{\frac{64\lambda}{15(2-\lambda)}}$, where c is a constant and λ is the smoothing parameter satisfying $0 < \lambda \le 1$.
Comments	The chart is constructed using the residuals from a fitted time series model. This chart plots one variable (M_i) in the chart. It indicates the source as well as direction of the shift when an out-of-control signal is issued. This chart is effective in detecting small shifts in the process mean and/or standard deviation.	The chart is constructed using the residuals from a fitted time series model. This chart plots one variable (M_i) in the chart. It indicates the source as well as direction of the shift when an out-of-control signal is issued. This chart is effective in detecting large shifts in the process mean and/or standard deviation.	This chart is sensitive in detecting small shifts in the process mean and/or variance. However, it is not effective for large shifts and at high levels of autocorrelation. It requires plotting two variables in the same chart, which is not a good feature particular if we take many samples.

Single Charts for Autocorrelated Processes Continued

Max-EWMA Chart for Autocorrelated Processes (MEWMAP Chart) (2007)

The residual at time t is $e_t = X_t - \xi_0 + \varphi(X_{t-1} - \xi_0) + \theta e_{t-1}$
The asymptotic mean and variance of the residuals after the shift are respectively

$$\mathrm{E}(e_t) = \frac{1-\varphi}{1-\theta}(\xi_1 - \xi_0)$$

$$\mathrm{Var}(e_t) = \sigma_{\gamma_0}^2 + \frac{\varphi^2 - 2\varphi\theta + 1}{1-\theta_0^2}(\sigma_{\varepsilon 1}^2 - \sigma_{\varepsilon 0}^2) + \frac{\sigma_{\alpha 1}^2 - \sigma_{\alpha 0}^2}{1-\theta_0^2}$$

$$\bar{\xi}_i = (\xi_{i1} + \xi_{i2} + \bullet\bullet\bullet + \xi_{in_i}) / n_i$$

$$MSE_i = \sum_{j=1}^{n_i} (\xi_{ij} - \bar{\xi}_i)^2 / n_i$$

$$Z_i = \sqrt{n}\frac{(\bar{\xi}_i - \xi_0)}{\sigma_{\gamma 0}}$$

$$Y_i = \Phi^{-1}\left\{H\left[\frac{(n)MSE_i}{\sigma_{\gamma 0}^2}; n\right]\right\}.$$

The EWMA statistics based on Z_i and Y_i are defined as:

$$U_i = (1-\lambda)U_{i-1} + \lambda Z_i,$$

$$V_i = (1-\lambda)V_{i-1} + \lambda Y_i$$

a new statistic for a new single control chart is defined as

$$M_i = \max\left[|U_i|,\ |V_i|\right]$$

$$\mathrm{E}(M_i) = \int_0^\infty y f(y;\sigma_{U_i})dy = 1.128379\sigma_{Ui}$$

$$\mathrm{Var}(M_i) = \int_0^\infty y^2 f(y;\sigma_{U_i})dy = 0.363381\sigma_{Ui}^2$$

$$\mathrm{UCL} = \sqrt{\frac{\lambda}{2-\lambda}}(1.128379 + 0.602810\, L)$$

The MEWMAP chart quickly detects moderate to large shifts in the mean and/or standard deviation at both low and high levels of autocorrelations than the Max-CUSUM chart for autocorrelated processes.

Let $\bar{X}_i = (X_{i1} + X_{i2} + \bullet\bullet\bullet + X_{in_i}) / n_i$ and $S_i^2 = \sum_{i=1}^{n_i} (X_{ij} - \bar{X}_i)^2 / (n_i - 1)$ be the mean and variance for the i^{th} sample respectively. Control charts for the process mean and standard deviation are based on $\bar{X}_i$ and S_i respectively. Tables I through V provide the summary statistics that are used to construct each of the charts discussed in this paper.

For the multivariate processes, the mean vector is $\boldsymbol{\mu}$ and the covariance matrix is $\boldsymbol{\Sigma}$. These parameters can be respectively estimated by $\bar{\mathbf{X}}$ and **S**, the sample mean vector and covariance matrix.

5. Conclusion

The single or simultaneous control charts are gaining popularity due to the fact that most of them are user friendly in terms of construction and clarity in showing the process parameter that has shifted. A general conclusion that can be drawn from a survey of these charts is that the single charts are more appealing than the individual charts. This is due to the fact that in addition to using one plotting variable in one chart, single charts are easy to interpret as they clearly show the source as well as the direction of the shift when an out-of-control signal is issued.

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