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A Numerical Method for Locating the Zeros of Ahlfors Map for Doubly Connected Regions

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Abstract

The Ahlfors map of an n-connected region is a n-to-one map from the region onto the unit disk. The Ahlfors map being n-to-one map has n zeros. Previously, the exact zeros of the Ahlfors map are known only for the annulus region. The zeros of the Ahlfors map for general bounded doubly connected regions has been unknown for many years. This paper presents a numerical method for computing the zeros of the Ahlfors map of any bounded doubly connected region. The method depends on the values of Szego kernel, its derivative and the derivative of boundary correspondence function of the Ahlfors map. The Ahlfors map and Szego kernel are both classically related to each other. Ahlfors map can be computed using Szego kernel without relying on the zeros of Ahlfors map. The Szego kernel is a solution of a Fredholm integral equation of the second kind with the Kerzman-Stein kernel. The numerical examples presented here prove the effectiveness of the proposed method.

Keywords: Ahlfors Map, Generalized Neumann Kernel, Neumann Kernel, Szego Kernel

1. Introduction

Ahlfors map is the conformal map from a multiply connected region onto the unit disk. If the region is simply connected then the Ahlfors map reduces to the Riemann map. Many of the geometrical features of a Riemann mapping function are shared with Ahlfors map. For a multiply connected region of connectivity $n > 1$, the Ahlfors map $f{:}\Omega \to D$ is analytic map that is onto, $f'(a) > 0$ and $f(a) = 0$, where a is fixed in Ω It maps Ω onto a unit disk D in n-to-one manner because it has $2n - 2$ branch points in the interior and is no longer one-to-one there.

Conformal mapping of multiply connected regions can be computed efficiently using the integral equation method. The integral equation method has been used by many authors to compute the one-to-one conformal mapping from multiply connected regions onto some standard canonical regions[3,4,6,8-11,13,15,19-23,27].

Some integral equations for Ahlfors map have been given in[1,7,14,16,17]. In[3], Kerzman and Stein have derived a uniquely solvable boundary integral equation for computing the Szego kernel of a bounded region and this method has been generalized in[1] to compute Ahlfors map of bounded multiply connected regions without relying on the zeros of Ahlfors map. In[7,14] the integral equations for Ahlfors map of doubly connected regions requires knowledge of zeros of Ahlfors map, which are unknown in general.

In this paper, we extend the approach of Sangawi[19,22-24] to construct an integral equation for solving $\theta'(t)$ where $\theta(t)$ is the boundary correspondence function of Ahlfors map of multiply connected region onto a unit disk.

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The plan of this paper is as follows: Section 2 presents some auxiliary materials. In Section 3, we present a formula for computing the derivative of Szego kernel which we used for finding the derivative of Ahlfors map giving another way of computing $\theta'(t)$ analytically. In Section 4, we present a method for computing the second zero of the Ahlfors map for doubly connected regions. In Section 5, we present some examples to illustrate our boundary integral equation method for finding the zeros of Ahlfors map for general doubly connected region. The numerical examples are first restricted to annulus region for which the exact Ahlfors map and its zeros are known is known, and then verified on general doubly connected region and obtained accurate results. Finally, Section 6 presents a short conclusion.

2. Auxiliary Materials

Let Ω be a bounded multiply connected region of connectivity $M + 1$. The boundary Γ consists of $M + 1$ smooth Jordan curves $\Gamma_0, \Gamma_1, ..., \Gamma_M$ such that $\Gamma_1, ..., \Gamma_M$ lie in the interior of Γ_0, where the outer curve Γ_0 has counterclockwise orientation and inner curves $\Gamma_1, ..., \Gamma_M$ have clockwise orientation. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1 \cup ... \cup \Gamma_M$ is usually that for which Ω is on the left as one traces the boundary as shown in Figure 1.

The curves Γ_j are parameterized by 2π periodic five times continously differentiable complex-valued functions $z_j(t)$ with non-vanishing first derivatives

$$z'_j(t) = dz_j(t)/\,dt \neq 0, \ \ t \in J_j = [0, 2\pi], \ \ j = 0, 1, ..., M$$

The total parameter domain J is defined as the disjoint union of M + 1 intervals $J_0, J_1, ..., J_M$.

The notation

$$z(t) = z_j(t), \ \ t \in J_j, \ \ j = 0, 1, ..., M. \tag{2.1}$$

is interpreted as follows[20]: For a given $\tilde{t} \in [0, 2\pi]$, to evaluate the value of $z(t)$ at $\tilde{t}$, we should know in advance the interval J_j to which $\tilde{t}$ belongs, i.e. we should know the boundary

Figure 1. A bounded multiply connected region of connectivity M + 1

Γ_j contains $z(\tilde{t})$, then we compute $z(\tilde{t}) = z_j(\tilde{t})$.

The generalized Neumann kernel formed with a complex continuously differentiable $2\pi-$ periodic function $A(t)$ for all is defined by[22,23]

$$\tilde{N}(t,s) = \frac{1}{\pi}\mathrm{Im}\left(\frac{A(t)}{A(s)}\frac{Z'(s)}{Z(s)-Z(t)}\right)$$

For $A(t) = 1$, $\tilde{N}(t, s)$ reduces to the well-known classical Neumann kernel $N(t, s)$. The kernel is continuous which takes on the diagonal the values with

$$\tilde{N}(t,t) = \frac{1}{\pi}\left(\frac{1}{2}\mathrm{Im}\frac{Z''(t)}{Z'(t)} - \mathrm{Im}\frac{A'(t)}{A(t)}\right)$$

It follows from the representation (2.2) that the adjoint krnel

$$N^*(s,t) = \tilde{N}(t,s) = \frac{1}{\pi}\mathrm{Im}\left(\frac{A(t)}{A(s)}\frac{Z'(s)}{Z(s)-Z(t)}\right),$$

can be represented as

$$N^*(s,t) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{\tilde{A}(s)}{\tilde{A}(t)}\frac{Z'(t)}{Z(t)-Z(s)}\right),$$

with the adjoint function is $\tilde{A}(t)$ is given by

$$\tilde{A}(t) = \frac{Z'(t)}{A(t)} \tag{2.3}$$

The generalized Neumann kernel $\tilde{N}(s, t)$ formed with $\tilde{A}(t)$ is given by

$$\tilde{N}(t,s) = \frac{1}{\pi}\mathrm{Im}\left(\frac{\tilde{A}(t)}{\tilde{A}(s)}\frac{Z'(s)}{Z(s)-Z(t)}\right),$$

which implies

$$\tilde{N}(s,t) = N^*(s,t).$$

It is also known that $\lambda = 1$is an eigenvalue of the kernel N with multiplicity 1 and $\lambda = -1$ is an eigenvalue of the kernel N with multiplicity M[23]. The eigenfunctions of N corresponding to the eigenvalue $\lambda = -1$ are $\{\chi^{[1]}, \chi^{[2]}, ..., \chi^{[M]}\}$ where

$$\chi^{[j]}(\zeta) = \begin{cases} 1, & \zeta \in \Gamma_j, \\ 0, & \text{otherwise, j=1,....}M \end{cases}$$

Let H^* be the space of all real Holder continuous $2\pi-$ periodic functions $\omega(t)$ of the parameter t on J_j for $j = 0, 1, ..., M$, *i.e.*

$$\omega(t) = \omega_j(t), \ \ t \in J_j \ \ j = 0, 1, ..., M.$$

We define the space by S

$$S = \text{span}\{\chi^{[1]}, \chi^{[2]}, ..., \chi^{[M]}\}$$

and the integral operators J by[13,15,24]

$$\mathrm{J}v = \frac{1}{2\pi}\sum_{j-0}^{M}\chi^{[f]}(t)(\chi^{[f]}, v),$$

with the inner product

$$(\chi^{[j]}, v) = \int_{J}\chi^{[j]}(s)V(s)ds.$$

We also define the Fredholm integral operator N^* by

$$N^*\,\psi(t) = \int_J N^*(t,s)\,\psi(s)\,ds, t\in J.$$

A complex-valued function $P(z)$ is said to satisfy the interior relationship if $P(z)$ is analytic in Ω and satisfies the non-homogeneous boundary relationship

$$P(z) = \frac{b(z)\overline{T(z)}}{\overline{G(z)}}\overline{P(z)} + \overline{H(z)}, \tag{2.5}$$

where $G(z)$ is analytic in Ω, Holder continuous on Γ, and $G(z) \neq 0$ on Γ. The boundary relationship (2.5) also has the following equivalent form

$$G(z) = \overline{b(z)}T(z)\frac{P(z)^2}{|P(z)|^2} + \frac{G(z)H(z)}{\overline{P(z)}}. \tag{2.6}$$

The following theorem gives an integral equation for an analytic function satisfying the interior non-homogeneous boundary relationship (2.5) or (2.6)

Theorem 2.2[14]**:** If the function $P(z)$ satisfies the interior relationship (2.5) or (2.6), then

$$T(z)P(z) + \int_{\Gamma}K(z,w)T(w)P(w)|dw|$$
$$+b(z)\left[\sum_{a_{j\,inside}\Gamma}\underset{W\text{-}a_j}{\text{Res}}\frac{P(w)}{(w-z)\,G(w)}\right]^{-} = -\overline{L_{\bar{R}}(Z)}, \tag{2.7}$$

where

$$K(z,w) = \begin{cases} \dfrac{1}{2\pi i}\left[\dfrac{T(z)}{z-w} - \dfrac{b(z)}{b(w)}\dfrac{\overline{T(w)}}{\bar{z}-\bar{w}}\right], & Z \neq w\in\Gamma, \\ -\dfrac{1}{2\pi i|z'(t)|}\left[\dfrac{P'(t)}{P(t)}\right], & where\ p(t) = b(z(t)), \\ & z = w\in\Gamma, \end{cases} \tag{2.8}$$

and

$$L_R^-(z) = \frac{-1}{2}\frac{H(z)}{T(z)} + \text{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{b(z)}H(w)}{b(w)(w-z)T(w)}dw \tag{2.9}$$

The symbol "–" in the superscript denotes the complex conjugate and the sum is over all those zeros that lie inside Ω. If G has no zeros in Ω, then the term containing the residue will not appear.

3. Integral Equation Method for Computing

Let $f(z)$ be the Ahlfors function which maps Ω conformally onto a unit disc. The mapping function f is determined up to a factor of modulus 1. The function f could be made unique by imposing the condition

$$f(a_j) = 0,\ f'(a_0) > 0,\ j = 0, 1, ..., M.$$

where $a_j \in \Omega$, $j = 0, 1, ..., M$ are the zeros of the Ahlfors map. The boundary values of f can be represented as

$$f(z_j(t)) = e^{i\theta_j(t)},\ \Gamma_j: z = z_j(t),\ 0 \le t \le \beta_j \tag{3.1}$$

where $\theta_j(t)$, $j = 0, 1, ..., M$ are the boundary correspondence functions of Γ_j. Also we have from (3.1) that

$$\frac{f'(z(t))z'(t)}{f(z(t))} = i\theta'(t). \tag{3.2}$$

The unit tangent to Γ at $z(t)$ is denoted by

$$T(z(t)) = \frac{z'(t)}{|z'(t)|}.$$

Thus it can be shown

$$f(z_j(t)) = \frac{1}{i}T(z_j(t))\frac{|\theta_j'(t)|\,f'(z_j(t))}{\theta_j'(t)\,|f'(z_j(t))|}, z_j\in\Gamma_j.$$

By the angle preserving property of conformal map, the image of Γ_0 remains in counter-clockwise orientation so $\theta'_0(t) > 0$, while the images of inner boundaries Γ_j in clockwise orientation so $\theta'_0(t) < 0$, for $j = 1, 2, ..., M$. The boundary relationship (3.1) can be written briefly as

$$f(z) = \text{sign}(\theta'(t))\frac{1}{i}T(z)\frac{f'(z)}{|f'(z)|}, z\in\Gamma. \tag{3.3}$$

Since the Ahlfors map can be written as

$$f(z) = \prod_{j=0}^{M}(z-a_j)\hat{g}(z),$$

Where $\hat{g}(z)$ is analytic in Ω and $\hat{g}(z) \neq 0$ in Ω Nazar et al.[9] have formulated an integral equation for computing θ' as

$$\theta'(t) + \int_J N(t,s)\,\theta'(s)\,ds = \phi(t) \tag{3.4}$$

where

$$\phi(t) = 2\,\mathrm{Im}\left[\sum_{j=0}^{M} \frac{z'(t)}{z(t)-a_j}\right], \tag{3.5}$$

$$N(z,w) = \begin{cases} \dfrac{1}{2\pi i}\left[\dfrac{T(z)}{z-w} - \dfrac{\overline{T(z)}}{\overline{z-w}}\right], & z \neq w \in \Gamma, \\ -\dfrac{1}{2\pi|z'(t)|}\mathrm{Im}\left(\dfrac{z''(t)}{z'(t)}\right), & z = w \in \Gamma. \end{cases}$$

The kernel $N(z, w)$ is the classical Neumann kernel. We want to find an alternate formula for $\theta'(t)$ which will enable us to apply (3.4) to find the zeros of Ahlfors map.

We next show how to compute $\theta'(t)$ from the knowledge of $S(z(t), a)$ and $S'(z(t), a)$ Note that the Ahlfors map is related to the Szego kernel $S(z, a_0)$ and the Garabedian kernel $L(z, a_0)$ by [2]

$$f(z) = \frac{s(z,a_0)}{L(z,a_0)}, z \in \Omega \cup \Gamma. \tag{3.6}$$

The Szego kernel $S(z,a_0)$ and Garabedian kernel $L(z,a_0)$ are related on Γ as

$$L(z,a_0) = -i\overline{T(z)s(z,a_0)}, \quad z \in \Gamma,$$

so that (3.6) becomes

$$f(z(t)) = \frac{1}{i}\frac{S(z(t),a_0)T(z(t))}{\overline{S(z(t)),(a_0)}}, z(t) \in \Gamma. \tag{3.7}$$

The Szego kernel is known to satisfy the integral equation[2]

$$s(z,a_0) + \int_\Gamma A(z,w)S(w,a_0)|dw| = g(z),\ z \in \Gamma \tag{3.8}$$

where

$$g(z(t)) = -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\overline{z(t)-a_0}},$$

$$A(z,w) = \begin{cases} \dfrac{1}{2\pi i}\left[\dfrac{T(w)}{z-w} - \dfrac{\overline{T(z)}}{\overline{z-w}}\right], & z \neq w \in \Gamma, \\ 0, & z=w \in \Gamma. \end{cases}$$

The kernel $A(z,w)$ is also known as the Kerzman-Stein kernel. Thus the boundary values of the Ahlfors map in (3.7) are completely determined from the boundary values of the Szego kernel.

With $z = z(t)$ and $w = z(s)$, (3.8) becomes

$$S(z(t),a_0) + \int_\Gamma A(z(t),z)S(z(s),a_0)|z'(s)|ds = g(z(t)). \tag{3.9}$$

Differentiate both sides with respect to t we get,

$$S'(z(t),a_0)z'(t) = g'(z(t))z'(t) - \int_J \left[\frac{d}{dt}A(z(t),z(s))z'(t)\right] S(z(s),a_0)|z'(s)|ds. \tag{3.10}$$

where

$$g'(z(t))z'(t) = \frac{1}{2\pi i}\left[\frac{\overline{T'(z(t))z'(t)}}{\overline{z(t)-a_0}} - \frac{\overline{z'(t)T(z(t))}}{\overline{(z(t)-a_0)^2}}\right] \tag{3.11}$$

$$T'(z(t))z'(t) = \frac{z''(t)}{2|z'(t)|} - \frac{z'^2(t)\overline{z''(t)}}{2|z'(t)|^3} \tag{3.12}$$

and

$$\frac{d}{dt}A(z(t),z(s)) = \begin{cases} \dfrac{1}{2\pi i}\left[\dfrac{-z'(t)T(z(s))}{(z(t)-z(s))^2} - \dfrac{\overline{T'(z(t))z'(t)}}{\overline{(z(t)-z(s))}} + \dfrac{\overline{T(z(t))z'(t)}}{\overline{(z(t)-z(s))^2}}\right], & t \neq s \in \Gamma, \\ \dfrac{1}{4\pi|z'(t)|}\left[\dfrac{1}{3}\mathrm{Im}\left(\dfrac{z'''(t)}{z'(t)}\right) - \mathrm{Re}\left(\dfrac{z''(t)}{z'(t)}\right)\mathrm{Im}\left(\dfrac{z''(t)}{z'(t)}\right)\right], & t = s \in \Gamma. \end{cases} \tag{3.13}$$

Using the solution of (3.9) and the values from (3.11)-(3.13), we can find the derivative of Szego kernel from (3.10). We next show how to find $\theta'(t)$ By defining the following terms,

$$f_p = f'(z(t))z'(t), S_p = S'(z(t),a_0)z'(t),$$
$$S_I = S(z(t),a_0), T_p = T'(z(t))z'(t), T_I = T(z(t)). \tag{3.14}$$

Differentiating both sides of (3.7) with respect to t we get

$$f_p = \frac{1}{i}\left[\frac{S_pT_2 + S_2T_p}{\overline{S_2}} - \frac{S_p(S_2T_2)}{\overline{(S_2)}^2}\right]. \tag{3.15}$$

From (3.2), we get

$$\theta'(t) = \frac{1}{i}\frac{f_p}{f}. \tag{3.16}$$

Using the values from (3.7) and (3.15) in (3.16), and after several manipulations, we get

$$\theta'(t) = 2\,\mathrm{Im}\left(\frac{S_p}{S_2}\right) + \mathrm{Im}\left(\frac{z''(t)}{z(t)}\right). \tag{3.17}$$

Which is the alternate formula for computing $\theta'(t)$.

4. Computing the Zeros of Ahlfors Map for Doubly Connected Regions

In Particular, if Ω is doubly connected region, i.e., $M = 1$ then (3.5) becomes

$$\phi(t) = 2\,\mathrm{Im}\left[\sum_{i=0}^{1} \frac{z'(t)}{z(t)-a_i}\right], \tag{4.1}$$

As ϕ is known from (3.4), and the zero $a_0 \in \Omega$ can be freely prescribed, the only unknown in (4.1) is the second zero a_1 of the Ahlfors map. The above equation can be written as

$$\mathrm{Im}\left[\frac{z'(t)}{z(t)-a_1}\right] = R_1(t), \tag{4.2}$$

where

$$R_1(t) = \frac{1}{2}\left[\phi - 2\,\mathrm{Im}\left(\frac{z'(t)}{z(t)-a_0}\right)\right]. \tag{4.3}$$

Now we suppose that

$$z(t) = x(t) + \mathrm{i}\,y(t), a_1 = \alpha_1 + i\beta_1.$$

Then becomes (4.2)

$$\frac{y'(t)(x(t)-\alpha_1) + x'(t)(\beta_1 - y(t))}{(x(t)-\alpha_1)^2 + (y(t)-\beta_1)^2} = R_1(t). \tag{4.4}$$

After some algebraic manipulations, we obtain

$$R_4(t) = -(R_2(t)\alpha_1 + R_3(t)\beta_1 + R_1(t)(\alpha_1^2 + \beta_1^2)), \tag{4.5}$$

where

$$R_2(t) = y'(t) - 2R_1(t)x(t),$$

$$R_3(t) = -2R_1(t)y(t) - x'(t),$$

$$R_4(t) = k_1(t)(x^2 + y^2) + x'(t)y(t) - y'(t)x(t), \tag{4.6}$$

Equation (4.5) can be viewed as containing two unknowns namely α_1 and β_1. At any two random values of t in the given interval, we can find the values of the coefficients R_1, $i = 1, \ldots, 4$. The 3D plot for $|f(z)|$ on the region, provides good initial guess. The MATHEMATICA command FindRoot solves the nonlinear equation (4.5) for the second zero of Ahlfors map

5. Numerical Examples

The obtained linear system from the discretized integral equation (3.9) is uniquely solvable for sufficiently large number of collocation points on each boundary component, since the integral equation (3.9) is uniquely solvable. By solving the integral equation (3.9) for $S(z(t),a_0)$ gives $S_p = S'(z(t),a_0)\, z'(t)$ from (3.10) and $\theta'(t)$ from (3.17). By (3.4) we get the values of ϕ and then from (4.5) we get the values of zero $a_1 = \alpha_1 + i\beta_1$. We then apply (3.7) and the Cauchy integral formula to compute $f(\alpha_1 + i\beta_1)$. For evaluating the Cauchy integral formula $f(z) = (1/(2\pi \mathrm{i}))\int_\Gamma f(w)/(w-z)dw$ numerically, we use the equivalent form

$$f(z) = \frac{\int_\Gamma \frac{f(w)}{w-z}dw}{\int_\Gamma \frac{1}{w-z}dw}, z \in \Omega, \tag{4.7}$$

which also works for $z \in \Omega$ near the boundary Γ. When the trapezoidal rule is applied to the integrals in (4.7), the term in the denominator compensates the error in the numerator[24]. All the computations were done using MATHEMATICA 10.0

Example 5.1
Consider an annulus region bounded by the two circles

$$\Gamma_0 : \{z(t) = e^{it}\},$$
$$\Gamma_1 : \{z(t) = re^{-it}\}, t : 0 \le t \le 2\pi, 0 < r < 1.$$

In[21], Tegtmeyer and Thomas computed the zeros of Ahlfors map for the annulus region as a_0 and where r is the radius of the inner circle. They have also considered the symmetry case where the zeros are $a_0 = \sqrt{r}$ and $a_1 = \sqrt{r}$. This example has also been considered in[22] where Ahlfors

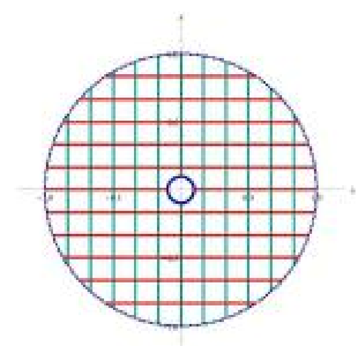

Figure. 2. The region for Example 5.1.

map was computed using a boundary integral equation related to a Riemann-Hilbert problem. Here we shall use these values of zeros of Ahlfors map for comparison with our proposed method for the annulus $r < |z| < 1$. In this example we have chosen $r = 0.1$ and the first zero $a_0 = 0.5$ See **Table 1** for numerical comparison of the computed second zeros a_{ln} from (4.5) and the exact second zeros of the Ahlfors map, $a_1 = \frac{r}{a_0}$. **Figures 3 and 4** show the 3D plots of $|f(z)|$ for both non-symmetric and symmetric cases for the region in Figure V using MATHEMATICA 10.0. The plot provides good initial guess for the second zeros of the Ahlfors map. This initial guess is then used in MATHEMATICA command to solve the nonlinear equation (4.5) for the second zero of Ahlfors map.

Example 5.2:

Consider a doubly connected region Ω bounded by Γ_0 and Γ_1 parameterized by

$$\Gamma_0 : \{z(t) = a_0\, e^{it} + b_0 e^{2it}\},$$
$$\Gamma_1 : \{z(t) = 0.2 + 0.2i + re^{-it}\}, t : 0 \le t \le 2\pi, 0 < r < 1.$$

Here $a_0 = 1$, $b_0 = 0.45$ and $r = 0.2$. The test region is shown in **Figure 5**. Given a first zero a_0 of the Ahlfors

Table 1. Absolute error $||a_{1n} - a_1||$ with $r = 0.1$ for Example 5.1.

Non-Symmetric case $\left(a_0 = 0.5, a_1 = -r\sqrt{a_0}\right)$	
n	$\|\|a_{1n} - a_1\|\|$
64	4.4064(–15)
Symmetric case $\left(a_0 = \sqrt{r}, a_1 = -\sqrt{r}\right)$	
n	$\|\|a_{1n} - a_1\|\|$
64	1.66533(–16)

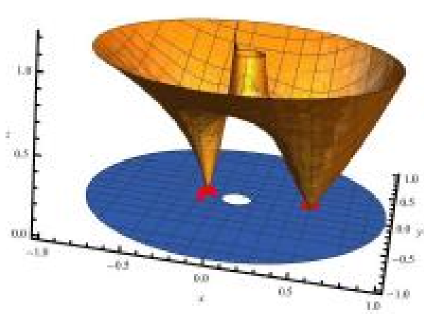

Figure. 3. Non-Symmetric-Case: 3D plot for |f(z)| for Example 5.1.

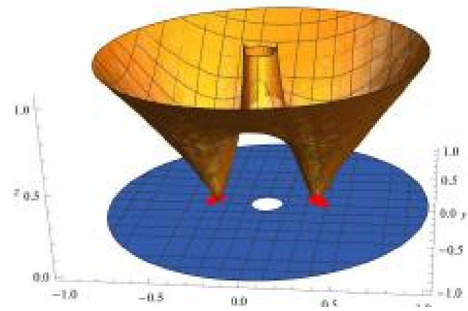

Figure. 4. Symmetric-Case: Absolute Ahlfors map for Example 5.1

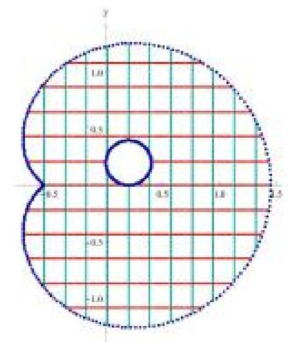

Figure. 5. The region for Example 5.2.

map, the exact second zero a_1 is unknown for this region. We compute a_{ln} from (4.5) which is the approximate value of a_1. The accuracy is measured by computing $f(a_{ln})$ from (4.7). See **Table 2** for the numerical values of a_{ln} and $f(a_{ln})$. **Figure 6** shows the 3D plot of $|f(z)|$ for the region in Figure 5 using MATHEMATICA.10.0. The plot provides good initial guess for the second zeros of the Ahlfors map. This initial guess is then used in MATHEMATICA command FindRoot to solve the nonlinear equation (4.5) for the second zero of Ahlfors map.

Example 5.3:

Consider a doubly connected region bounded by Γ_0 and Γ_1 parameterized by

$$\Gamma_0 : \{z(t) = (\cos(t) + i\sin(t))\sqrt{1-(1-r_1^2)\cos^2(t)}\},$$
$$\Gamma_1 : \{z(t) = r_2(\cos(t) - i\sin(t))\}, t : 0 \le t \le 2\pi, 0 < r < 1.$$

Here $r_1 = 0.5$, $r_2 = 0.2$. The test region is shown in **Figure 7.** Given a first zero a_0 of the Ahlfors map, the exact second zero a_1 is also unknown for this region. We compute a_{ln} from (24) which the approximate value of is a_1. The accuracy is measured by computing $f(a_{ln})$ from (4.7). See **Table 3** for the numerical values of a_{ln} and $f(a_{ln})$. **Figure** 8 shows the 3D plot of $|f(z)|$ for the region in

Table 2. Numerical values of a_{1n} and $f(a_{1n})$ for Example 5.2.

$a_0 = 0.5 + 0.5i$		
n	a_{1n}	$f(a_{1n})$
64	−0.2175289249 − 0.070255296i	4.81616(−06)
128	−0.217531264 − 0.0702542624i	1.27221(−05)
256	−0.2175312637 − 0.070254262i	6.30004(−08)

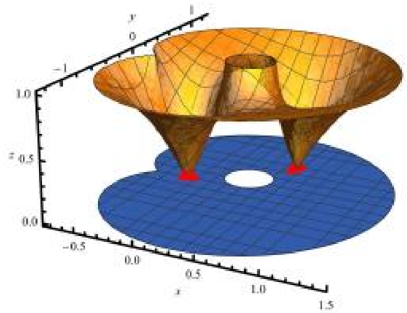

Figure. 6. 3D plot for |f(z)| for Example 5.2

Table 3. Numerical values of a_{1n} and $f(a_{1n})$ for Example 5.3.

i		
n	a_{1n}	$f(a_{1n})$
64	−0.2643924458 − 0.07971385519i	1.16172(−04)
128	−0.26439967046 − 0.07970158266i	6.98872(−10)
256	−0.264399670402 − 0.079701582722i	6.65912(−15)

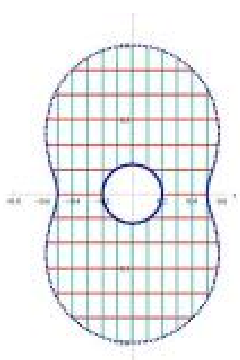

Figure. 7. The region for Example 5.3

Figure 7 using MATHEMATICA10.0. The plot provides good initial guess for the second zeros of the Ahlfors map. This initial guess is then used in MATHEMATICA command FindRoot to solve the nonlinear equation (4.5) for the second zero of Ahlfors map.

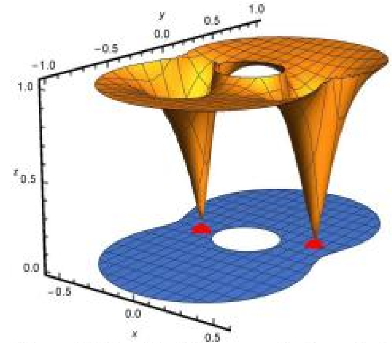

Figure. 8. 3D plot for |f(z)| for Example 5.3

6. Conclusion

In this paper, we have constructed a numerical method for finding the zeros of the Ahlfors map of doubly connected regions. We derived two formulas for the derivative of the boundary correspondence function $\theta(t)$ of the Ahlfors map, and the derivative of the Szego kernel function $S(z,a_0)$These formulas were then used to find the zeros of the Ahlfors map for any smooth doubly connected regions. Analytical method for computing the exact zeros of Ahlfors map for annulus region and a particular triply connected regions are presented in[20] and[21] but the problem of finding zeros for arbitrary doubly connected regions is the first time presented in this paper. The numerical examples presented have illustrated that our method is reliable and has high accuracy.

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