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Quadratic Stochastic Operators on Segment [0,1] and Their Limit Behavior

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Abstract

In this paper, we construct some class of quadratic stochastic operator defined on continual state space [0, 1], and show that such operators are regular transformation.

Keywords: Continual State Space, Quadratic Stochastic Operator, Regular Transformation

1. Introduction

Quadratic stochastic operator was first introduced in Bernstein's work[1] and it was considered an important source of analysis for the study of dynamical properties and modeling in various fields such as biology[2–5], physics[6], game theory[7–10], etc.

A quadratic stochastic operator acts on the set of all probability measures on measurable space (X, F), where X is a state space and F is σ-algebra on X. The theory of quadratic stochastic operators is well developed for the case when state space X is a finite set[1–17]. In[18–24] the authors studied quadratic stochastic operators defined on countable or continual state space X.

Let us first recall some well known notions and notations. Let X is a state space, F is σ-algebra on X, (X, F) be a measurable space, and $S(X, F)$ be the set of all probability measures on (X, F). It is known that the set $S(X, F)$ is a compact, convex space and a form of Dirac measure δ_x which defined by:

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{1}$$

for any $A \in F$ are extremal elements of $S(X, F)$.

Let $\{P(x, y, A) : x, y \in X, A \in F\}$, be a family of functions $P(x, y, A) : X \times X \times F \to R$ which satisfy the following conditions:

(i) $P(x, y, .) \in S(X, F)$ for any fixed $x, y \in X$, that is, $P(x, y, .) : F \to [0, 1]$ is the probabilistic measure on F,
(ii) $P(x, y, A)$ is measurable function on $(X \times X, F \otimes F)$ which regarded as a function of two variables x and y with fixed $A \in F$,
(iii) $P(x, y, A) = P(y, x, A)$ for any $x, y \in X$ and $A \in F$.

Definition 1: A mapping $V: S(X, F) \to S(X, F)$ is called a quadratic stochastic operator generated by the family of functions $\{P(x, y, A) : x, y \in X, A \in F\}$ if for an arbitrary measure $\lambda \in S(X, F)$, then the measure $\lambda' = V\lambda$ is defined as follows:

$$\lambda'(A) = \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y) \tag{2}$$

where $A \in F$ is an arbitrary measurable set.

Assume $\{V^k(\lambda) \in S(X, F) : k = 0, 1, 2, \ldots\}$ is a trajectory of the initial measure $\lambda \in S(X, F)$, where $V^{k+1}(\lambda) = V(V^k(\lambda))$ for all $k = 0, 1, 2, \ldots$, with $V^0(\lambda) = \lambda$.

Definition 2: A measure $\lambda \in S(X, F)$ is called a fixed point of a quadratic stochastic operator V, if $V(\lambda) = \lambda$.

Definition 3: A quadratic stochastic operators V is called a *regular*, if for any initial measure $\lambda \in S(X, F)$ there exist a *strong limit:*

$$\lim_{n\to\infty} V^n(\lambda) = \mu \tag{3}$$

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i.e., for any measurable set $A \in F$ there exists limit:

$$\lim_{n\to\infty} V^n(\lambda)(A) = \mu(A) \tag{4}$$

where $\mu \in S(X, F)$.

Note that the limit point is a fixed point of a quadratic stochastic operator V. Thus, the fixed points of quadratic stochastic operator describe a limit or long run behaviour of the trajectories at any initial point. Limit behaviour of the trajectories and the fixed points of quadratic stochastic operator have been studied in many applied problems[1–17].

If $X = \{0, 1, 2, \ldots, n\}$, a natural σ-algebra F on X is a power set $P(X)$, and the set of all probability measures on (X, F) has the following form:

$S(X, F) \equiv S^{n-1} = \left\{x = (x_1, \ldots, x_n) \in R^n : x_i \geq 0, \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} x_i = 1\right\}$, then a measure $P(i, j, .)$ is a discrete measure on finite set X and one can define this measure by specifying measure of each singleton as follows $P(i, j, \{k\}) \equiv P_{ij,k}$.

A definition of quadratic stochastic operator on finite X is presented as follows.

Definition 4: A mapping $V: S^{n-1} \to S^{n-1}$ is called a quadratic stochastic operator, if for any

$x = (x_1, x_2, \ldots, x_n) \in S^{n-1}$, Vx is defined as

$$(Vx)_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j \tag{5}$$

where the coefficients $P_{ij,k}$ which also called as coefficients of heredity satisfy the following conditions:

(i) $P_{ij,k} \geq 0$, (ii) $P_{ij,k} = P_{ij,k}$, (iii) $\sum_{k=1}^{n} P_{ij,k} = 1$,

for all $i, j, k = \{1, 2, \ldots, n\}$. It is evident that these three conditions fully consistent with conditions formulated in general case.

Definition 5: The quadratic stochastic operator V is called Volterra, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$.

The biological view of Volterra quadratic stochastic operators is rather clear, which the offspring repeats one of its parents. Evidently, for any Volterra quadratic stochastic operator,

$$P_{ik,i} + P_{ki,k} = P_{ik,i} + P_{ik,k} = 1 \tag{6}$$

for all l $i, k = 1, \ldots, n$; $i \neq k$.

The trajectory behavior of Volterra quadratic stochastic operators with finite state space X have been studied in many publications[8,11–15,17]. For the case of infinite state space X, papers[18,19] are discussing about infinite dimensional Volterra quadratic stochastic operators. Furthermore, in[20–23], the authors introduced and studied Poisson, Geometric, and Gaussian quadratic stochastic operators on infinite state space.

In[24], the authors introduced a quadratic stochastic operator on segment [0, 1] and proved that this operator is a regular transformation.

In this paper, we generalize a construction of a quadratic stochastic operator generated by 2-partition on segment [0, 1].

2. A Quadratic Stochastic Operator Generated by 2-Partition ξ

Let (X,F) be a measurable space with continual state space X.

Definition 6: A probabilistic measure μ on (X, F) is said to be *discrete***,** if there exists a finitely many elements $\{x_1, x_2, \ldots, x_n\} \subset X$, such that $\mu(\{x_i\}) = p_i$ for $i = 1, \ldots, n$, with $\sum_{i=1}^{n} p_i = 1$. Then, $\mu(X \backslash \{\{x_1, x_2, \ldots, x_n\}\}) = 0$ and for any $A \in F$, $\mu(A) = \sum_{x_i \in A} \mu(x_i)$.

Recall that a partition of (X,F) is a disjoint collection of elements of F whose union is X. We shall be interested in finite partitions. They will be denoted as $\xi = \{A_1, \ldots, A_k\}$ and is called measurable k-partition.

Let $\xi = \{A_1, A_k\}$ be a measurable 2-partition of the segment of the segment $X = [0, 1]$ where $A_1 = \left[0, \frac{1}{a}\right)$ and $A_2 = \left[\frac{1}{a}, 1\right]$, and $\xi = \{B_1, B_2, B_3\}$ be a corresponding partition of the unit square $X \times X = [0, 1] \times [0, 1]$, where $B_1 = A_1 \times A_1$, $B_2 = A_2 \times A_2$ and $B_3 = A_1 \times A_2 \cup A_2 \times A_1$. Note that the consideration of this partition is stipulated by the condition $P(y, x, .) = P(x, y, .)$.

We define the family $\{P(x, y, .) : y \in [0, 1]\}$ of discrete probability measures on (X, F) as follows: if $x, y \in B_k$, where $k = 1, 2, 3$, then

(i) for $x < y$ assume $P(x, y, \{x\}) = p_k$ and $P(x, y, \{y\}) = q_k$, (7)

(ii) for $x = y$ assume $P(x, x, \{x\}) = 1,$ (8)
(iii) for $x < y$ assume $P(y, x, .) = P(x, y, .),$ (9)

where $p_k + q_k = 1$ and $p_k, q_k \geq 0$ for all $k = 1, 2, 3$.

Let V be an operator generated by family of functions (7–9). Note that this operator is a natural generalization of Volterra quadratic stochastic operator[4–7].

Remark Note that $\alpha = 1$ then we have operator considered in[24].

Below we investigate limit behavior of the trajectory $\{V^k(\lambda) \in S(X, F) : k = 0, 1, 2, \ldots\}$, where $V^{k+1}(\lambda) = V(V^k(\lambda))$ for all $k = 0, 1, 2, \ldots$, with $V^0(\lambda) = \lambda$.

2.1 Discrete Initial Measure λ

If initial measure λ is discrete, one can show that the sequence $\{V^k(\lambda)\}$ converges to a Dirac δ measure.

Theorem 1: Let V be a quadratic stochastic operator generated by family of functions (7)–(9). If initial measure λ is a discrete, then the sequence $\{V^k(\lambda)\}$ converges to a Dirac δ measure.

Proof Let a measure λ is a convex linear combination of two Dirac measures, i.e., $\lambda = \alpha\delta_a + (1 - \alpha)\delta_b$, where $\alpha \in [0, 1]$ and $a, b \in [0, 1]$.

Simple algebra gives $\lambda'(a) = \alpha^2 + 2p_3\alpha(1 - \alpha)$ and $\lambda'(b) = (1 - \alpha)^2 + 2q_3\alpha(1 - \alpha)$,

i.e., $\lambda' = (\alpha^2 + 2p_3\alpha(1-\alpha)\delta_a + ((1-\alpha)^2 + 2q_3\alpha(1-\alpha))\delta_b$.

It is easy to show that the sequence $\{V^k(\lambda)\}$ converges to Dirac measure δ_a if $p_3 > \frac{1}{\alpha}$, and Dirac measure δ_b if $p_3 < \frac{1}{\alpha}$.

Similarly, one can show that for any initial discrete measure λ, the sequence $\{V^k(\lambda)\}$ converges to a Dirac δ measure.

2.2 Continuous Initial Measure λ

For the case of continuous initial measure λ, let $\lambda \in S(X, F)$ be a continuous probability measure,

$$A = [a, b] \in F, A \subset [0, 1], A_1 = \left[0, \frac{1}{\alpha}\right) \text{ and } A_2 = \left[\frac{1}{\alpha}, 1\right],$$

where $\alpha \in \mathbb{N}$. Then, we can consider the following two cases:

(i) $A \subset A_1$, and
(ii) $A \subset A_2$.

For case (i) $A \subset A_1$ with $A^C = [0, a) \cup (b, 1]$, we have

$$\lambda'(A) = \int_0^1 \int_0^1 P(x, y, A)d\lambda(x)d\lambda(y)$$

$$= \int_a^b \int_a^b 1 \cdot d\lambda(x)d\lambda(y) + \int_0^a \int_a^b p_1 \cdot d\lambda(x)d\lambda(y) + \int_a^b \int_b^{\frac{1}{\alpha}} q_1 \cdot d\lambda(x)d\lambda(y)$$

$$+ \int_a^b \int_{\frac{1}{\alpha}}^1 q_3 \cdot d\lambda(x)d\lambda(y) + \int_a^b \int_0^a p_1 \cdot d\lambda(x)d\lambda(y) + \int_b^{\frac{1}{\alpha}} \int_a^b q_1 \cdot d\lambda(x)d\lambda(y)$$

$$+ \int_{\frac{1}{\alpha}}^1 \int_a^b q_1 \cdot d\lambda(x)d\lambda(y) + \int_0^a \int_0^a 0 \cdot d\lambda(x)d\lambda(y) + \int_b^1 \int_0^a 0 \cdot d\lambda(x)d\lambda(y)$$

$$+ \int_0^a \int_b^1 0 \cdot d\lambda(x)d\lambda(y) + \int_b^1 \int_b^1 0 \cdot d\lambda(x)d\lambda(y)$$

$$= \lambda^2([a,b]) + p_1\lambda([a,b])\lambda([0,a)) + q_1\lambda((b,\frac{1}{\alpha}])\lambda([a,b])$$

$$+ q_3\lambda([\frac{1}{\alpha},1])\lambda([a,b]) + p_1\lambda([0,a])\lambda((a,b]) + q_1\lambda([a,b])\lambda([b,\frac{1}{\alpha}])$$

$$+ p_1\lambda([0,a])\lambda((a,b]) + q_1\lambda([a,b])\lambda([b,\frac{1}{\alpha}]) + q_3\lambda([a,b])\lambda((\frac{1}{\alpha},1])$$

$$= \lambda([a,b])\left[\lambda([a,b]) + 2p_1\lambda([0,a)) + 2q_1\lambda((b,\frac{1}{\alpha}]) + 2q_3\lambda((\frac{1}{\alpha},1])\right]$$

and

$$\lambda''(A) = \lambda'([a,b])\left[\lambda'([a,b]) + 2p_1\lambda'([0,a)) + 2q_1\lambda'((b,\frac{1}{\alpha}]) + 2q_3\lambda'((\frac{1}{\alpha},1])\right]$$

$$= \lambda([a,b])\left[\lambda([a,b]) + 2p_1\lambda([0,a)) + 2q_1\lambda((b,\frac{1}{\alpha}]) + 2q_3\lambda((\frac{1}{\alpha},1])\right]$$

$$\cdot\left\{\lambda([a,b])\left[\lambda([a,b]) + 2p_1\lambda([0,a)) + 2q_1\lambda((b,\frac{1}{\alpha}]) + 2q_3\lambda((\frac{1}{\alpha},1])\right]\right.$$

$$+2p_1\lambda([0,a])\left[\lambda([0,a]) + 2q_1\lambda([a,\frac{1}{\alpha})) + 2q_3\lambda((\frac{1}{\alpha},1])\right]$$

$$+2q_1\lambda((b,\frac{1}{\alpha}])\left[\lambda((b,\frac{1}{\alpha}]) + 2p_1\lambda([0,b)) + 2q_3\lambda([\frac{1}{\alpha},1])\right]$$

$$\left.+2q_3\lambda([\frac{1}{\alpha},1])\left[\lambda([\frac{1}{\alpha},1]) + 2p_3\lambda([0,\frac{1}{\alpha}])\right]\right\}.$$

For case (ii) $A \subset A_2$ with $A^C = [0, a) \cup (b, 1]$, we have

$$\lambda'(A) = \int_0^1 \int_0^1 P(x, y, A)d\lambda(x)d\lambda(y)$$

$$= \int_a^b\int_a^b 1\cdot d\lambda(x)d\lambda(y) + \int_0^{\frac{1}{a}}\int_a^b p_3 \cdot d\lambda(x)d\lambda(y) + \int_{\frac{1}{a}}^a\int_a^b p_2 \cdot d\lambda(x)d\lambda(y)$$

$$+ \int_a^b\int_b^1 q_2 \cdot d\lambda(x)d\lambda(y) + \int_a^b\int_0^{\frac{1}{a}} p_3 \cdot d\lambda(x)d\lambda(y) + \int_b^a\int_{\frac{1}{a}}^a p_2 \cdot d\lambda(x)d\lambda(y)$$

$$+ \int_b^1\int_a^b q_2 \cdot d\lambda(x)d\lambda(y) + \int_0^a\int_0^a 0\cdot d\lambda(x)d\lambda(y) + \int_b^1\int_0^a 0\cdot d\lambda(x)d\lambda(y)$$

$$+ \int_0^a\int_b^1 0\cdot d\lambda(x)d\lambda(y) + \int_b^1\int_b^1 0\cdot d\lambda(x)d\lambda(y)$$

$$= \lambda^2([a,b]) + p_3\lambda([a,b])\lambda([0,\frac{1}{a})) + p_2\lambda((a,b])\lambda([\frac{1}{a},a])$$

$$+ q_2\lambda([b,1])\lambda([a,b]) + p_3\lambda([0,\frac{1}{a}])\lambda((a,b])$$

$$+ p_2\lambda([\frac{1}{a},a])\lambda([a,b]) + q_2\lambda([a,b])\lambda((b,1])$$

$$= \lambda([a,b])\left[\lambda([a,b]) + 2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda((\frac{1}{a},a]) + 2q_2\lambda((b,1])\right]$$

and

$$\lambda''(A) = \lambda'([a,b])\left[\lambda'([a,b]) + 2p_3\lambda'([0,\frac{1}{a})) + 2p_2\lambda'((\frac{1}{a},a]) + 2q_2\lambda'((b,1])\right]$$

$$= \lambda([a,b])\left[\lambda([a,b]) + 2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda((\frac{1}{a},a]) + 2q_2\lambda((b,1])\right].$$

$$\left\{\lambda([a,b])\left[\lambda([a,b]) + 2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda((\frac{1}{a},a]) + 2q_2\lambda((b,1])\right]\right.$$

$$+2p_3\lambda([0,\frac{1}{a}])\left[\lambda([0,\frac{1}{a}]) + 2q_3\lambda((\frac{1}{a},1])\right]$$

$$+2p_2\lambda([\frac{1}{a},a))\left[\lambda([\frac{1}{a},a)) + 2p_3\lambda([0,\frac{1}{a}])\right]$$

$$\left.+2q_2\lambda((b,1])\left[\lambda((b,1]) + 2p_3\lambda([0,\frac{1}{a}]) + 2p_2\lambda([\frac{1}{a},b))\right]\right\}.$$

It is evident that the measure $\lambda' = V\lambda$ is absolutely continuous with respect to measure λ. Then according Radon-Nikodym Theorem, there exists non-negative measurable function $f_\lambda : X \to R$ called density, such that

$$\lambda'(A) = \int_A f_\lambda(x)d\lambda(x).$$

For $x \le \frac{1}{a}$, the derivations of the density function are presented as follows. For rather small segment $[x, x + \Delta x]$, we have,

$$\lambda'([x,x+\Delta x]) = \lambda([x,x+\Delta x])\left[\lambda([x,x+\Delta x]) + 2p_1\lambda([0,x))\right.$$

$$\left.+2q_1\lambda((x+\Delta x,\frac{1}{a}]) + 2q_3\lambda((\frac{1}{a},1])\right] \quad (10)$$

and

$$f_\lambda(x) = \lim_{\Delta x\to 0}\frac{\lambda'([x,x+\Delta x])}{\lambda([x,x+\Delta x])}$$

$$= \lim_{\Delta x\to 0}\left[\lambda([x,x+\Delta x]) + 2p_1\lambda([0,x))\right.$$

$$\left.+2q_1\lambda((x+\Delta x,\frac{1}{a}]) + 2q_3\lambda((\frac{1}{a},1])\right]$$

$$= 2p_1\lambda([0,x)) + 2q_1\lambda((x,\frac{1}{a}]) + 2q_3\lambda((\frac{1}{a},1]) \text{ if } x \le \frac{1}{a}.$$

Respectively, for $x > \frac{1}{a}$, the derivations of the density function are presented as follows. For rather small segment $[x, x + \Delta x]$, we have,

$$\lambda'([x,x+\Delta x]) = \lambda([x,x+\Delta x])\left[\lambda([x,x+\Delta x])\right.$$

$$\left.+2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda((\frac{1}{a},a]) + 2q_2\lambda((b,1])\right] \quad (11)$$

and

$$f_\lambda(x) = \lim_{\Delta x\to 0}\frac{\lambda'([x,x+\Delta x])}{\lambda([x,x+\Delta x])}$$

$$= \lim_{\Delta x\to 0}\left[\lambda([x,x+\Delta x]) + 2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda((\frac{1}{a},a]) + 2q_2\lambda((b,1])\right]$$

$$= 2p_3\lambda([0,\frac{1}{a})) + 2p_2\lambda([\frac{1}{a},x)) + 2q_2\lambda((x,1]) \text{ if } x > \frac{1}{a}$$

Therefore,

$$f_\lambda(x)=\begin{cases}2p_1\lambda([0,x))+2q_1\lambda((x,\frac{1}{a}])+2q_3\lambda((\frac{1}{a},1]) & \text{if } x\le\frac{1}{a},\\ 2p_3\lambda([0,\frac{1}{a}))+2p_2\lambda([\frac{1}{a},x))+2q_2\lambda((x,1]) & \text{if } x>\frac{1}{a}.\end{cases}$$

Now, consider a measure $\lambda''=V\lambda'$. It is evident that,

$$\lambda''(A)=\int_A f_{\lambda'}(x)d\lambda'(x)$$

and since λ'' is absolutely continuous with respect to measure λ, then:

$$\lambda''(A)=\int_A F_\lambda^{(2)}(x)d\lambda(x).$$

We find implicit form of the function $F_\lambda^{(2)}(x)$. For $x\le\frac{1}{a}$, according to (10), we have,

$$\lambda''([x,x+\Delta x])=\lambda'([x,x+\Delta x])\Big[\lambda'([x,x+\Delta x])+2p_1\lambda'([0,x)) +2q_1\lambda'((x+\Delta x,\frac{1}{a}])+2q_3\lambda'([\frac{1}{a},1])\Big]$$

$$=\lambda([x,x+\Delta x])\Big[\lambda([x,x+\Delta x])+2p_1\lambda([0,x)) +2q_1\lambda((x+\Delta x,\frac{1}{a}])+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$\cdot\Big\{\lambda([x,x+\Delta x])\Big[\lambda([x,x+\Delta x])+2p_1\lambda([0,x)) +2q_1\lambda((x+\Delta x,\frac{1}{a}])++2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2p_1\lambda[0,x])\Big[\lambda([0,x])+2q_1\lambda((x,\frac{1}{a}])+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2q_1\lambda((x+\Delta x,\frac{1}{a}])\Big[\lambda((x+\Delta x,\frac{1}{a}])+2p_1\lambda((0,x+\Delta x)) +2q_3\lambda([\frac{1}{a},1])\Big]+2q_3\lambda([\frac{1}{a},1])\Big[\lambda([\frac{1}{a},1])+2p_3\lambda([0,\frac{1}{a}])\Big]\Big\}.$$

Then,

$$F_\lambda^{(2)}(x)=\Big[2p_1\lambda([0,x))+2q_1\lambda((x,\frac{1}{a}])++2q_3\lambda([\frac{1}{a},1])\Big]$$

$$\cdot\Big\{2p_1\lambda[0,x])\Big[\lambda([0,x])+2q_1\lambda((x,\frac{1}{a}])+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2q_1\lambda((x,\frac{1}{a}])\Big[\lambda((x,\frac{1}{a}])+2p_1\lambda((0,x))+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2q_3\lambda([\frac{1}{a},1])\Big[\lambda([\frac{1}{a},1])+2p_3\lambda([0,\frac{1}{a}])\Big]\Big\}\ \textit{if } x\le\frac{1}{a}.$$

Respectively, for $x>\frac{1}{a}$, according to (11), we have:

$$\lambda''([x,x+\Delta x])=\lambda'([x,x+\Delta x])\Big[\lambda'([x,x+\Delta x])+2p_3\lambda'([0,\frac{1}{a}]) +2p_2\lambda'([\frac{1}{a},x))+2q_2\lambda'((x+\Delta x,1])\Big]$$

$$=\lambda([x,x+\Delta x])\Big[\lambda([x,x+\Delta x])+2p_3\lambda([0,\frac{1}{a}]) +2p_2\lambda([\frac{1}{a},x))+2q_2\lambda((x+\Delta x,1])\Big]$$

$$\cdot\Big\{\lambda([x,x+\Delta x])\Big[\lambda([x,x+\Delta x])+2p_3\lambda([0,\frac{1}{a}])+2p_2\lambda([\frac{1}{a},x)) +2q_2\lambda((x+\Delta x,1])\Big]$$

$$+2p_3\lambda[0,\frac{1}{a}])\Big[\lambda([0,\frac{1}{a}])+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2p_2\lambda([\frac{1}{a},x))\Big[\lambda([\frac{1}{a},x))+2p_3\lambda([0,\frac{1}{a}])+2q_2\lambda((x,1])\Big]$$

$$+2q_2\lambda((x+\Delta x,1])\Big[\lambda((x+\Delta x,1])+2p_3\lambda([0,\frac{1}{a}])+2p_2\lambda([\frac{1}{a},x+\Delta x))\Big]\Big\}$$

Then,

$$F_\lambda^{(2)}(x)=\Big[2p_3\lambda([0,\frac{1}{a}])+2p_2\lambda([\frac{1}{a},x))+2q_2\lambda((x,1])\Big]\cdot$$

$$\Big\{2p_3\lambda[0,\frac{1}{a}])\Big[\lambda([0,\frac{1}{a}])+2q_3\lambda([\frac{1}{a},1])\Big]$$

$$+2p_2\lambda([\frac{1}{a},x))\Big[\lambda([\frac{1}{a},x))+2p_3\lambda([0,\frac{1}{a}])+2q_2\lambda((x,1])\Big]$$

$$+2q_2\lambda((x,1])\Big[\lambda((x,1])+2p_3\lambda([0,\frac{1}{a}])+2p_2\lambda([\frac{1}{a},x))\Big]\Big\}\ \text{if } x>\frac{1}{a}$$

Similarly, one can show that a measure $V^n(\lambda)$ is absolutely continuous with respect to λ for any n and

$$(V^n\lambda)(A)=\int_A F_\lambda^{(n)}(x)d\lambda(x)$$

The computations for $F_\lambda^{(n)}(x)$ are shown as follows:

Let

$$g_\lambda^{(1)}(x)=\lambda([0,x]),\ h_\lambda^{(1)}(x)=\lambda\left(\left[x,\frac{1}{a}\right]\right),\ p_\lambda^{(1)}=\lambda\left(\left[0,\frac{1}{a}\right]\right)$$

for $x\le\frac{1}{a}$,

$$\varphi_\lambda^{(1)}(x)=\lambda\left(\left[\frac{1}{a},x\right]\right),\ \psi_\lambda^{(1)}(x)=\lambda([x,1]),\ q_\lambda^{(1)}=\lambda\left(\left[\frac{1}{a},1\right]\right)$$

for $x>\frac{1}{a}$,

and

$$g_\lambda^{(n+1)}(x)=g_\lambda^{(n)}(x)\left[g_\lambda^{(n)}(x)+2q_1h_\lambda^{(n)}(x)+2q_3q_\lambda^{(n)}\right],$$

$$h_\lambda^{(n+1)}(x)=h_\lambda^{(n)}(x)\left[h_\lambda^{(n)}(x)+2p_1g_\lambda^{(n)}(x)+2q_3q_\lambda^{(n)}\right],$$

$$q_\lambda^{(n+1)}=q_\lambda^{(n)}\left[q_\lambda^{(n)}+2p_3p_\lambda^{(n)}\right],$$

$$p_\lambda^{(n+1)}=p_\lambda^{(n)}\left[p_\lambda^{(n)}+2q_3q_\lambda^{(n)}\right],$$

$$\varphi_\lambda^{(n+1)}(x)=\varphi_\lambda^{(n)}(x)\left[\varphi_\lambda^{(n)}(x)+2q_2\psi_\lambda^{(n)}(x)+2p_3p_\lambda^{(n)}\right],$$

$$\psi_\lambda^{(n+1)}(x)=\psi_\lambda^{(n)}(x)\left[\psi_\lambda^{(n)}(x)+2p_2\varphi_\lambda^{(n)}(x)+2p_3p_\lambda^{(n)}\right],$$

with $p_\lambda^{(n+1)}+q_\lambda^{(n+1)}=1$ for every n = 1, 2, 3.

Then, $$f_\lambda^{(n)}(x)=\begin{cases}2p_1g_\lambda^{(n)}(x)+2q_1h_\lambda^{(n)}(x)+2q_3q_\lambda^{(n)} & \text{if } x\le\frac{1}{a},\\ 2p_2\varphi_\lambda^{(n)}(x)+2q_2\psi_\lambda^{(n)}(x)+2p_3p_\lambda^{(n)} & \text{if } x>\frac{1}{a}.\end{cases}$$

Hence,

$$F_\lambda^{(1)}(x)=f_\lambda^{(1)}(x),\text{ and } F_\lambda^{(2)}(x)=f_\lambda^{(1)}(x)f_\lambda^{(2)}(x).$$

Using induction, one can show that for any n we have:

$$F_\lambda^{(n)}(x)=\prod_{i=1}^{n}f_\lambda^{(i)}(x).$$

If $\lambda=m$ is a usual Lebesgue measure on [0,1], then:

$$g_m^{(1)}(x)=m([0,x])=x,\ h_m^{(1)}(x)=m\left(\left[x,\frac{1}{a}\right]\right)=\frac{1}{a}-x$$

$$p_m^{(1)}=m\left(\left[0,\frac{1}{a}\right]\right)=\frac{1}{a},\ \varphi_m^{(1)}(x)=m\left(\left[\frac{1}{a},x\right]\right)=x-\frac{1}{a},$$

$$\psi_m^{(1)}(x)=m([x,1])=1-x,\ q_m^{(1)}=m\left(\left[\frac{1}{a},1\right]\right)=\frac{1}{a},$$

and one can compute implicit form of $F_m^{(n)}(x)$ for any n, where:

$$f_m^{(1)}(x)=\begin{cases}2p_1x+2q_1\left(\frac{1}{a}-x\right)+q_3 & \text{if } x\le\frac{1}{a},\\ 2p_2\left(x-\frac{1}{a}\right)+2q_2(1-x)+p_3 & \text{if } x>\frac{1}{a}.\end{cases}$$

In general, for any n the function $F_\lambda^{(n)}(x)$ is a piecewise monotone function on $\left[0,\frac{1}{a}\right)$ and $\left(\frac{1}{a},1\right]$ respectively and has a point of discontinuity at $x=\frac{1}{a}$ with $F_\lambda^{(n)}\left(\frac{1}{a}^-\right)\ne F_\lambda^{(n)}\left(\frac{1}{a}^+\right)$ where $F_\lambda^{(n)}\left(\frac{1}{a}^-\right)=\lim_{x\to\frac{1}{a}^-}F_\lambda^{(n)}x$ and $F_\lambda^{(n)}\left(\frac{1}{a}^+\right)=\lim_{x\to\frac{1}{a}^+}F_\lambda^{(n)}x$.

If $\lambda=m$, one can show numerically that:

(i) $F^{(n)}(0)\to\infty$, $F^{(n)}(\frac{1}{a}^-)\to0$ $F^{(n)}(\frac{1}{a}^+)\to0$, $F^{(n)}(1)\to0$ if $p_1<\frac{1}{a}$ and $p_3<\frac{1}{a}$, (12)

(ii) $F^{(n)}(0)\to0, F^{(n)}(\frac{1}{a}^-)\to\infty$ $F^{(n)}(\frac{1}{a}^+)\to0, F^{(n)}(1)\to0$ if $p_1>\frac{1}{a}$ and $p_3<\frac{1}{a}$, (13)

(iii) $F^{(n)}(0)\to0, F^{(n)}(\frac{1}{a}^-)\to0$ $F^{(n)}(\frac{1}{a}^+)\to\infty, F^{(n)}(1)\to0$ if $p_2<\frac{1}{a}$ and $p_3>\frac{1}{a}$, (14)

and

$F^{(n)}(0)\to0$, $F^{(n)}(\frac{1}{a}^-)\to0$ $F^{(n)}(\frac{1}{a}^+)\to0$, $F^{(n)}(1)\to\infty$ if $p_2>\frac{1}{a}$ and $p_3>\frac{1}{a}$. (15)

Let us consider the following 8 cases:

(i) $p_1<\frac{1}{a}, p_2<\frac{1}{a}, p_3<\frac{1}{a}$,

(ii) $p_1<\frac{1}{a}, p_2<\frac{1}{a}, p_3>\frac{1}{a}$,

(iii) $p_1<\frac{1}{a}, p_2>\frac{1}{a}, p_3<\frac{1}{a}$,

(iv) $p_1 < \frac{1}{a}, p_2 > \frac{1}{a}, p_3 > \frac{1}{a},$

(v) $p_1 > \frac{1}{a}, p_2 < \frac{1}{a}, p_3 < \frac{1}{a},$

(vi) $p_1 > \frac{1}{a}, p_2 < \frac{1}{a}, p_3 > \frac{1}{a},$

(vii) $p_1 > \frac{1}{a}, p_2 > \frac{1}{a}, p_3 < \frac{1}{a},$

(viii) $p_1 > \frac{1}{a}, p_2 > \frac{1}{a}, p_3 > \frac{1}{a},$

and for some fixed values of these parameters we plot the graph of functions $\left\{F_m^{(n)}(x)\right\}$.

Below we show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$, $n = 1, 2, \ldots$ converges to some fixed point for arbitrary values of parameters p_1, p_2, p_3, where the rate of convergence depends on these parameters. If the values of parameters p_1, p_2, p_3, , close to 0 or 1 then the sequence of functions $\left\{F_m^{(n)}(x)\right\}$, $n = 1, 2, \ldots$ converges very fast (exponentially) and if the values of parameters p_1, p_2, p_3, close to $\frac{1}{a}$ then the sequence of functions $\left\{F_m^{(n)}(x)\right\}$, $n =$ 1, 2, … converges very slowly. In each case, we consider two variants of a: Variant (a) if $a = 4$ for parameters p_1, p_2, p_3 close to 0 or 1; Variant (b) if $a = 10$ for parameters p_1, p_2, p_3 close to 0 or 1.

We plot graph of functions $\left\{F_m^{(n)}(x)\right\}$ with $n = 1, 2, \ldots$ 10, where curve $F_m^{(1)}(x)$ is red in colour, while $F_m^{(2)}(x)$ is black, $F_m^{(3)}(x)$ is green, $F_m^{(4)}(x)$ is blue, $F_m^{(5)}(x)$ is yellow, $F_m^{(6)}(x)$ is pink, $F_m^{(7)}(x)$ is orange, $F_m^{(8)}(x)$ is purple, $F_m^{(9)}(x)$ is brown, and lastly $F_m^{(10)}(x)$ is grey colour.

Figure 1 presents graph of functions $\left\{F_m^{(n)}(x)\right\}$ in the first case where $p_1 < \frac{1}{a}, p_2 < \frac{1}{a}, p_3 < \frac{1}{a}$.

Figure 1 (a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = 0$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure δ_0.

Now consider the second case where $p_1 < \frac{1}{a}$, $p_2 < \frac{1}{a}$, $p_3 > \frac{1}{a}$.

Figure 2(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = \frac{1}{a}$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure $\delta_{\frac{1}{a}}$.

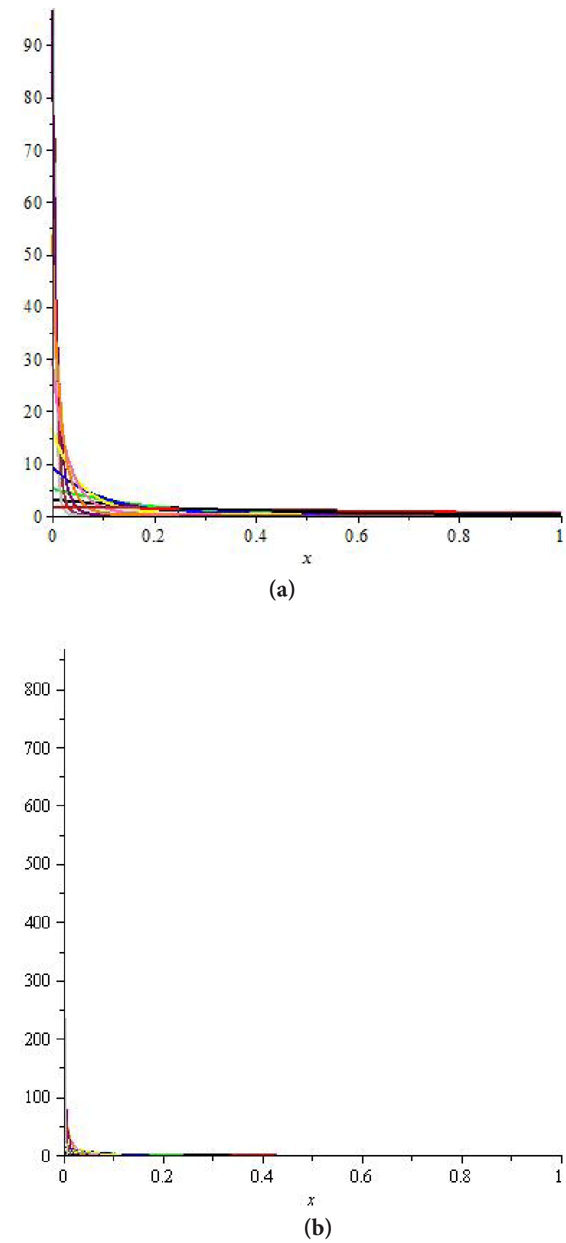
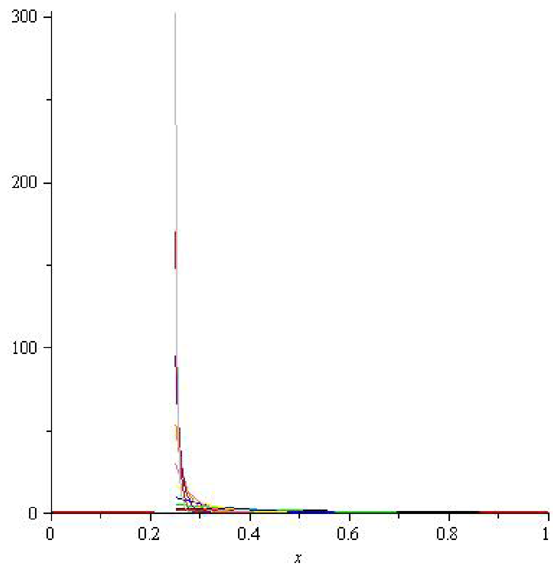
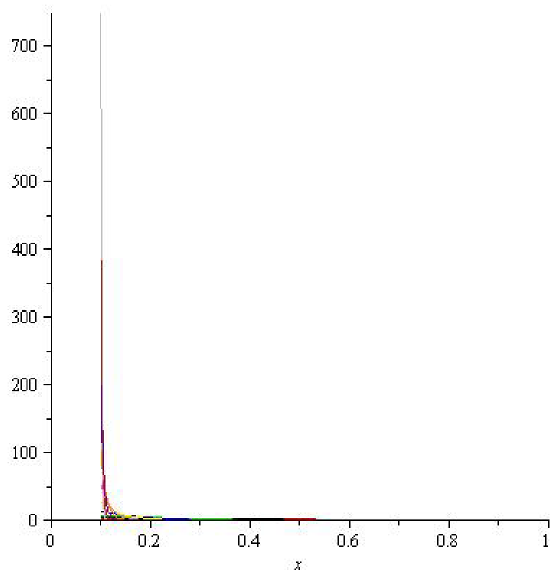


Figure 1. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 4$, $p_1 = 0.1, p_2 = 0.2, p_3 = 0.15$. **(b)**. Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 4$, $p_1 = 0.01$, $p_2 = 0.02$, $p_3 = 0.03$.

Then, we consider the third case where $p_1 < \frac{1}{a}$, $p_2 > \frac{1}{a}$, $p_3 < \frac{1}{a}$.



(a)

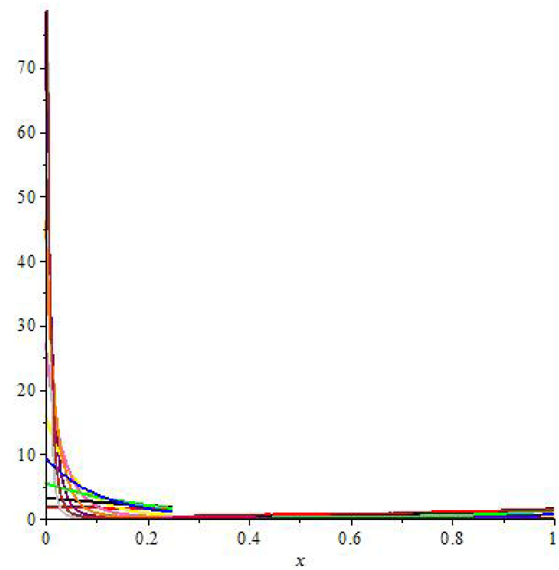


(b)

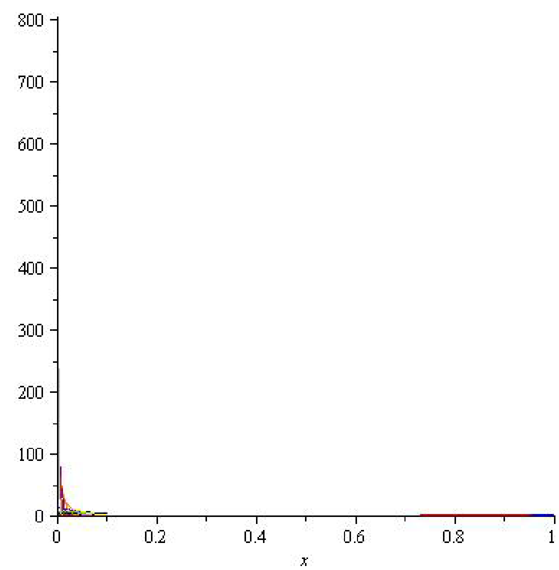
Figure 2. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when a = 4, $p_1 = 0.15, p_2 = 0.11, p_3 = 0.8$. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when a = 10, p_1 = 0.05, p_2 = 0.03, p_3 = 0.9.

Figure 3(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = 0$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure δ_0.

Let us consider the fourth case where $p_1 < \frac{1}{a}$, $p_2 > \frac{1}{a}$, $p_3 > \frac{1}{a}$.



(a)

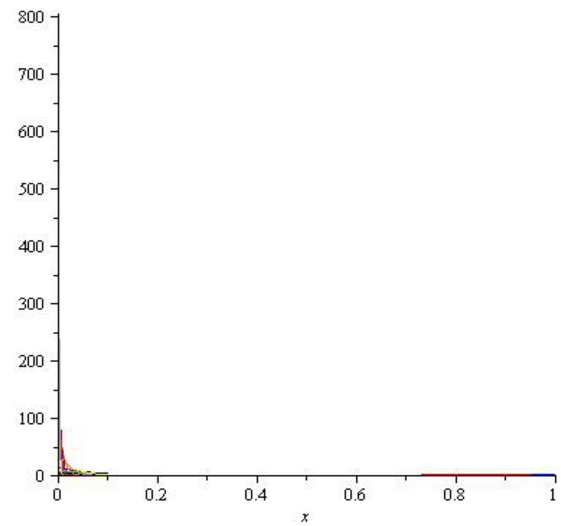


(b)

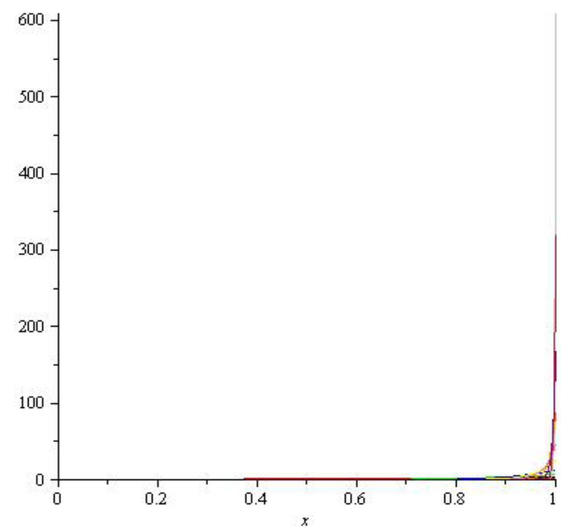
Figure 3. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when a = 4, p_1 = 0.15, p_2 = 0.85, p_3 = 0.1. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when a = 10, p_1 = 0.03, p_2 = 0.85, p_3 = 0.1.

Figure 4(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point x = 1 and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure δ_1.

Next, we consider the fifth case where $p_1 > \frac{1}{a}$, $p_2 < \frac{1}{a}$, $p_3 < \frac{1}{a}$.

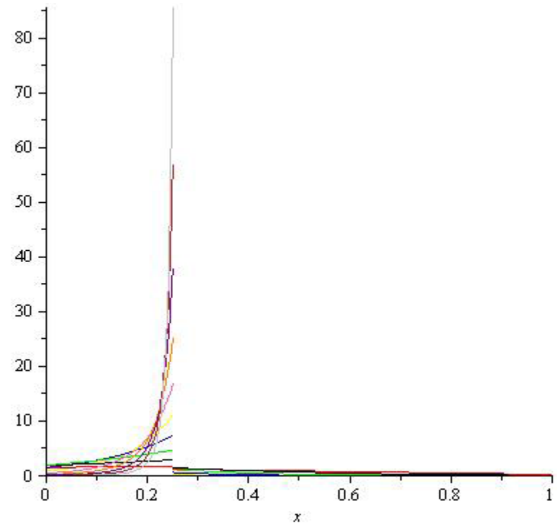


(a)

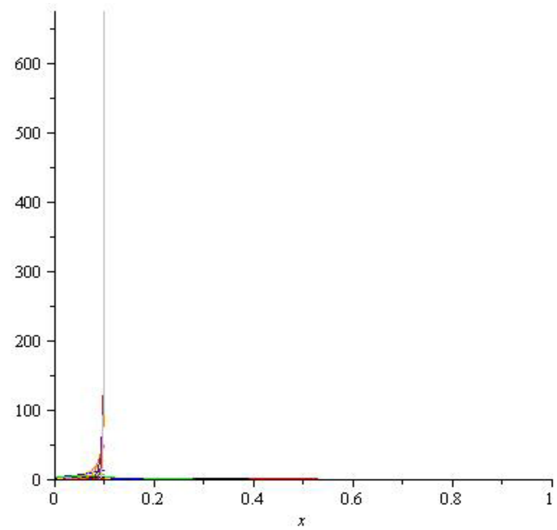


(b)

Figure 4. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $\alpha = 4$, $p_1 = 0.15, p_2 = 0.85, p_3 = 0.9$. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $\alpha = 10$, $p_1 = 0.05$, $p_2 = 0.95$, $p_3 = 0.9$.



(a)



(b)

Figure 5. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $\alpha = 4$, $p_1 = 0.75, p_2 = 0.15, p_3 = 0.1$. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $\alpha = 10$, $p_1 = 0.95$, $p_2 = 0.06$, $p_3 = 0.02$.

Figure 5(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = \frac{1}{\alpha}$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure $\delta_{\frac{1}{\alpha}}$.

Now we consider the sixth case where $p_1 > \frac{1}{\alpha}$, $p_2 < \frac{1}{\alpha}$, $p_3 > \frac{1}{\alpha}$.

Figure 6(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = \frac{1}{\alpha}$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure $\delta_{\frac{1}{\alpha}}$.

Now we consider the seventh case where $p_1 > \frac{1}{\alpha}$, $p_2 > \frac{1}{\alpha}$, $p_3 < \frac{1}{\alpha}$.

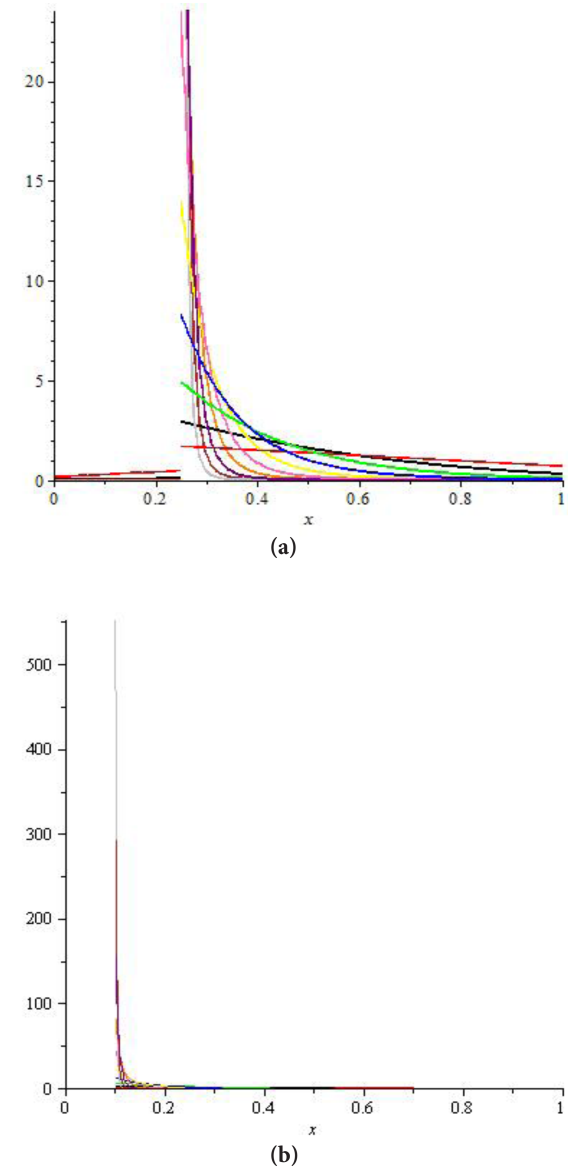


Figure 6. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 4$, $p_1 = 0.75$, $p_2 = 0.75$, $p_3 = 0.85$. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 10$, $p_1 = 0.82$, $p_2 = 0.06$, $p_3 = 0.95$.

Figure 7(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = \frac{1}{a}$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure $\delta_{\frac{1}{a}}$.

Lastly, we consider the eighth case where $p_1 > \frac{1}{a}$, $p_2 > \frac{1}{a}$, $p_3 > \frac{1}{a}$.

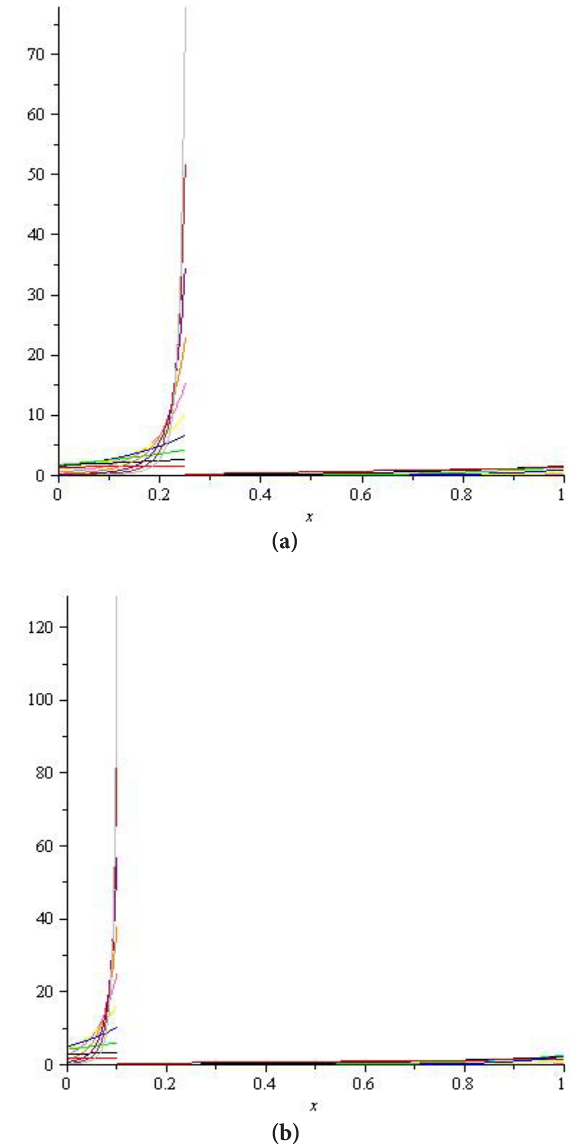


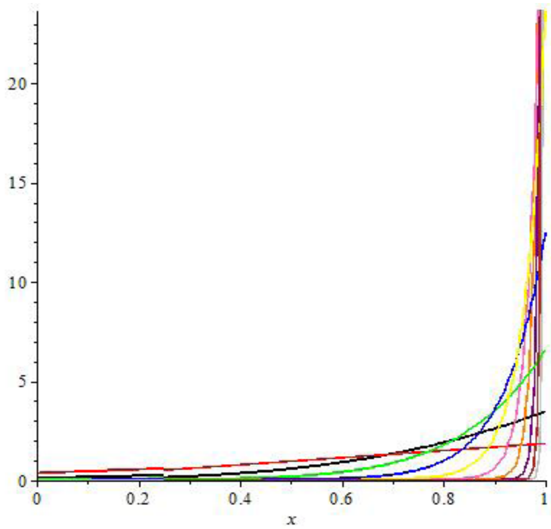
Figure 7. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 4$, $p_1 = 0.75$, $p_2 = 0.85$, $p_3 = 0.15$. **(b)**. Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 10$, $p_1 = 0.75$, $p_2 = 0.85$, $p_3 = 0.05$.

Figure 8(a) and (b) show that the sequence of functions $\left\{F_m^{(n)}(x)\right\}$ converges to δ-Dirac function at point $x = 1$ and respectively the sequence of measures $\{V^k(m)\}$ converges to Dirac measure δ_1.

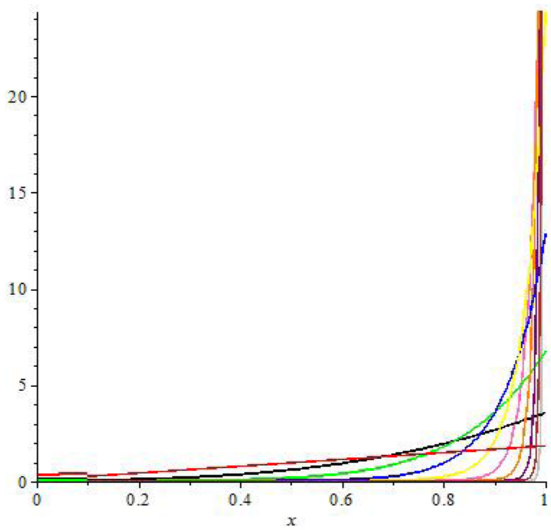
Thus, different a will give the same Dirac measure depends on the parameters p_1, p_2, p_3.

Therefore, we have proved the following theorem.

Theorem 2: Let V is a quadratic stochastic operator generated by family of functions (7)-(9). Then for any ini-



(a)



(b)

Figure 8. **(a)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 4$, $p_1 = 0.75$, $p_2 = 0.95$, $p_3 = 0.85$. **(b)** Graph of Functions $\left\{F_m^{(n)}(x)\right\}$ when $a = 10$, $p_1 = 0.75$, $p_2 = 0.95$, $p_3 = 0.85$.

tial continuous measure $\lambda \in S(X, F)$ there exist a strong limit of the sequence of measures $\{V^k(m)\}$ where:

(i) $\lim_{n\to\infty} V^n(\lambda) = \delta_0$ if $p_1 < \frac{1}{a}$ and $p_3 < \frac{1}{a}$,

(ii) $\lim_{n\to\infty} V^n(\lambda) = \delta_{\frac{1}{a}^-}$ if $p_1 > \frac{1}{a}$ and $p_3 < \frac{1}{a}$,

(iii) $\lim_{n\to\infty} V^n(\lambda) = \delta_{\frac{1}{a}^+}$ if $p_2 < \frac{1}{a}$ and $p_3 > \frac{1}{a}$,

and

(iv) $\lim_{n\to\infty} V^n(\lambda) = \delta_1$ if $p_2 > \frac{1}{a}$ and $p_3 > \frac{1}{a}$.

3. Conclusion

A limit behavior of quadratic stochastic operator V generated by arbitrary 2-partition ξ is the regular transformation.

4. Acknowledgements

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