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Conservation Laws and Hamiltonian Symmetries of Whitham-Broer-Kaup Equations

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Abstract

In the present paper, conservation laws of the tri-Hamiltonain system of equations Whitham-Broer-Kaup (WBK) are investigated by applying the first homotopy formula. Hamiltonian symmetries of the system are constructed by using the corresponding Hamiltonian operators and the conserved densities.

Keywords: Equations Whitham-Broer-Kaup, Conservation Laws, Hamiltonian Symmetry

1. Introduction

The system of equations Whitham-Broer-Kaup (WBK)

$$u_t = uu_x + v_x - \frac{1}{2}u_{x^2},\ _t = (uv)_x + \frac{1}{2}v_{x^2}, \qquad (1)$$

is equivalent under a change of variables, to a system of Boussinesq equations modelling the bi-directional propagation of long waves in shallow water, first found by Whitham[3], Broer[2] and Kaup[4].

Lie symmetry analysis for the WBK equations was derived by Z. Zhang, X. Yong and Y. Chen[12]. In addition, they found conservation laws of the system of equations WBK by means of scaling symmetry. Kupershmidt[5], showed that the physical system is integrable by inverse scattering and subsequently showed that this system is tri-Hamiltonian.

Our paper is organized as follows. In section 2, we analyze the three Hamiltonian structures associated with the Hamiltonian description of the system of equations WBK and probe Jacobi identity as well as compatibility of the three structures using the method of functional multi-vectors. In section 3, we construct conservation laws of the system by considering multipliers of order three. In section 4, we deduce Hamiltonian symmetries of the WBK equations from the Hamiltonian operators and conserved densities. In section 5, we summarize our results.

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2. Hamiltonian Operators

In this section, we will provide the background definitions and results in Hamiltonian operators that will be used along this paper. Much of it is stated as in[9].

Let $x = (x^1, \cdots, x^p)$ denote the spatial variables, and $u = (u^1, \cdots, u^q)$ the field variables (dependent variables), so each u^α is a function of $x^1, \cdots, x^p$ and the time t. We will be concerned with autonomous systems of evolution equations

$$u_t = K[u], \qquad (2)$$

in which $K[u] = (K_1[u], \cdots, K_q[u])$ is a q-tuple of differential functions, where the square brackets indicate that each K_α is a function of x, u and finitely many partial derivatives of each u^α with respect to $x^1, \cdots, x^p$. A system of evolution equations is said to be Hamiltonian if it can be written in the form

$$u_t = \mathrm{D}.\mathrm{E}_u(H), \qquad (3)$$

Here $\mathrm{H}[u] = \int H[u]dx$ is the Hamiltonian functional, and Hamiltonian function $H[u]$ depends on x, u, and the derivatives of the u's with respect to the x's; $\mathrm{E}_u = (\mathrm{E}_1, \cdots, \mathrm{E}_q)$ denotes the Euler operator or variational derivative with respect to u. The Hamiltonian operator D is a $q \times q$ matrix differential operator, which may depend on both x, u, and derivatives of u (but not on t), and is required to be (formally) skew-adjoint relative to the L^2-inner product

$< f, g >= \int f.g dx = \int \sum f^{\alpha}.g^{\alpha} dx$, so $D^* = -D$, where $*$ denotes the formal L^2 adjoint of a differential operator. In addition, D must satisfy a nonlinear "Jacobi condition" that the corresponding poisson bracket

$$\{P, Q\} = \int E_u[P].DE_u[Q] dx, \tag{4}$$

$$P = \int P[u] dx, \quad Q = \int Q[u] dx,$$

satisfies the Jacobi identity. In the spatial case that D is a field-independent skew-adjoint differential operator, meaning that the coefficient of D do not depend on u or its derivatives (but may depend on x), the Jacobi conditions are automatically satisfied; For more general field-dependent operators, the complicated Jacobi conditions can be considerably simplified by the "functional multi-vector" method which is described in detail in[9]. Multi-vectors are the dual objects of differential forms. To preserve the notational distinction between the two, we use the notation θ_J^{α} for the uni-vector corresponding to the one-form du_J^{α}; thus a vertical multi-vector is a finite sum of terms, each of which is the product of a differential function times a wedge product of the basic uni-vectors. The space of functional multi-vectors is the cokernel of the total divergence, so that two vertical multi-vectors determine the same functional multi-vector if and only if they differ by a total divergence. The functional multi-vector determined by $\breve{\Theta}$ is denoted, suggestively by an integral sign: $\Theta = \int \breve{\Theta} dx$. In particular, $\int \breve{\Theta} dx = 0$ if and only if $\breve{\Theta} = \mathrm{Div} \breve{\Psi}$ for some vertical multi-vector $\breve{\Psi}$. This implies that we can integrate functional multi-vectors by parts:

$$\int \breve{\Theta} \wedge (D_i \breve{\Psi}) dx = -\int (D_i \breve{\Theta}) \wedge \breve{\Psi} dx. \tag{5}$$

The principal example of a functional bi-vector is that determined by a Hamiltonian differential operator D, which is

$$\Theta_D = \int \theta \wedge D(\theta) dx. \tag{6}$$

Finally, define the formal prolonged vector field

$$\mathbf{pr v}_{D\theta} = \sum_{\alpha, J} D_J \left(\sum_{\beta} D_{\alpha\beta} \theta^{\beta} \right) \frac{\partial}{\partial u_J^{\alpha}}, \tag{7}$$

which acts on differential functions to produce uni-vectors. We further let $\mathbf{pr v}_D \theta$ act on vertical multi-vectors by wedging the result of its action on the coefficient differential functions with the product of the θ's. Since $\mathbf{pr v}_D \theta$ commutes with the total derivative, there is also a well-defined action of $\mathbf{pr v}_D \theta$ on the space of functional multi-vectors, which essentially amounts to bringing it under the integral sign.

By virtue of the following theorem, one can determine whether or not a differential operator is genuinely Hamiltonian. Let D be a skew-adjoint differential operator with corresponding bi-vector Θ_D as above. Then D is a Hamiltonian operator if and only if

$$\mathbf{pr v}_{D\theta}(\Theta_D) = 0. \tag{8}$$

The proof that (8) is equivalent to the Jacobi identity for the poisson bracket determined by D can be found in (9).

The system of equations WBK admits three Hamiltonian operators

$$D_0 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 2D_x & D_x.u - D_x^2 \\ uD_x + D_x^2 & 2vD_x + v_x \end{pmatrix}, \tag{9}$$

$$D_2 = \begin{pmatrix} 4uD_x + 2u_x & 4vD_x + 2v_x + D_x(D_x - u)^2 \\ 4vD_x + 2v_x + (D_x + u)^2 D_x & (D_x + u)(2vD_x + v_x) - (2vD_x + v_x)(D_x - u) \end{pmatrix},$$

and so can be written in Hamiltonian form in three distinct ways,[5]. The skew symmetry of these Hamiltonian structures is manifest. The Proof of the Jacobi identiy for this structures as well their compatibility can be shown through the standard method of functional multi vectors. Since the coefficients of the operator D_0 do not depend on u or its derivatives, then D_0 is automatically a Hamiltonian operator. For the operator D_1 it is sufficient to prove that $\mathbf{pr v}_{D_1\theta}(\Theta_{D_1}) = 0$, where Θ_{D_1} is the corresponding functional bi-vector and $\theta = (\theta, \zeta)$ s.t θ and ζ are the basic uni-vectors corresponding to u and v respectively.

We can construct the bi-vector associated with the structure D_1 as

$$\Theta_{D_1} = \frac{1}{2} \int \{\theta \wedge D_1 \theta\} dx \tag{10}$$

$$= \frac{1}{2} \int \{2\theta \wedge \theta_x + u\theta \wedge \zeta_x + u_x \theta \wedge \zeta - \theta \wedge \zeta_{xx}$$

$$+ u\zeta \wedge \theta_x + \zeta \wedge \theta_{xx} + 2v\zeta \wedge \zeta_x + v_x \zeta \wedge \zeta\} dx$$

So, by using integration by parts

$$\Theta_{D_1} = \frac{1}{2} \int \{2\theta \wedge \theta_x + 2u\zeta \wedge \theta_x - \theta \wedge \zeta_{xx} + \zeta \wedge \theta_{xx} + 2v\zeta \wedge \zeta_x\} dx \tag{11}$$

In continuation, we apply prolongation relations in order to prove the Jacobi identity.

$$\mathrm{pr}\mathbf{v}_{D_1\theta}(u) = 2\theta_x + u\zeta_x + u_x\zeta - \zeta_{xx}, \tag{12}$$

$$\mathrm{pr}\mathbf{v}_{D_1\theta}(v) = u\theta_x + \theta_{xx} + 2v\zeta_x + v_x\zeta, \tag{13}$$

$$\mathrm{pr}\mathbf{v}_{D_1\theta}(\Theta_{D_1}) = \int\{(2\theta_x + u\zeta_x + u_x\zeta - \zeta_{xx}) \tag{14}$$

$$\wedge\zeta\wedge\theta_x + (u\theta_x + \theta_{xx} + 2v\zeta_x + v_x\zeta)\wedge\zeta\wedge\zeta_x\}\,dx$$

again by using integration by parts, relation (8) for the operator D_1 is obtained. Thus, D_1 is a Hamiltonian operator. Similarly, the operator D_2 is Hamiltonian. Therefore, the system of equations WBK is tri-Hamiltonian, meaning that it can be written as a Hamiltonian system using any one of the three Hamiltonian operators. Concentrating on the simpler Hamiltonian operators D_0 and D_1, the corresponding Hamiltonian functionals are respectively

$$H_0 = \int -\frac{1}{2}u_x v + \frac{1}{2}u^2 v + \frac{1}{2}v^2 dx, \tag{15}$$

$$H_1 = \int \frac{1}{2}uv \tag{16}$$

Furthermore, any two of these operators form a Hamiltonian pair, namely, not only are D_i and D_j, $i, j = 1, \cdots, 3$ genuine Hamiltonian structures, any arbitrary linear combination of them is as well. In the following, we show that the Hamiltonian operators D_0 and D_1 form a Hamiltonian pair. So, it is sufficient to prove that

$$\mathbf{pr}\mathbf{v}_{D_0\theta}(\Theta_{D_1}) + \mathbf{pr}\mathbf{v}_{D_1\theta}(\Theta_{D_0}) = 0, \tag{17}$$

where Θ_{D_0} and Θ_{D_1} are the bivectors corresponding to the Hamiltonian structures D_0 and D_1, respectively. Since D_0 has constant coefficients, $\mathrm{pr}\mathbf{v}_{D_1\theta}(\Theta_{D_0}) = 0$.

So we only need to verify $\mathrm{pr}\mathbf{v}_{D_1\theta}(\Theta_{D_0}) = 0$ where $\mathrm{pr}\mathbf{v}_{D_0\theta}(u) = \zeta_x$, $\mathrm{pr}\mathbf{v}_{D_0\theta}(v) = \theta_x$. Hence,

$$\mathrm{pr}\mathbf{v}_{D_0\theta}(\Theta_{D_1}) = \int\{\zeta\wedge\theta_x\wedge\zeta_x + \theta_x\wedge\zeta\wedge\zeta_x\}\,dx = 0 \tag{18}$$

In a similar way, Hamiltonian operators D_0, D_2 and D_1, D_2 are compatible. As a result we can state the following proposition: The system of equations WBK admits three Hamiltonian operators D_0, D_1, D_2 and so it can be written as a Hamiltonian system using any one of the three Hamiltonian operators. In addition, any two of these operators form a Hamiltonian pair.

3. Conservation Laws

Consider a system of N partial differential equations of order n with p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, given by

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, N. \tag{19}$$

A conservation law of a PDE system (19) is a divergence expression

$$D_1 P_1 + \ldots + D_p P_p = 0 \tag{20}$$

holding for all solutions $u = f(x)$ of the given system. In (20), $P_i(x, u^{(r)}), i = 1, \ldots, p$, are called the fluxes of the conservation law, and the highest-order derivative r present in the fluxes is called the order of the conservation law.

If one of the independent variables of PDE system (19) is time t, the conservation law (20) takes the form

$$D_t T + \mathrm{Div} X = 0 \tag{21}$$

where Div is the spatial divergence of X with respect to the spatial variables $x = (x^1, \ldots, x^p)$. Here T is referred to as a density, and $X = (X_1, \ldots, X_p)$ as spatial fluxes of the conservation law (20). The conserved density, T, and the associated flux, $X = (X_1, \ldots, X_p)$, are functions of x, t, u and the derivatives of u with respect to both x and t.

In particular, every admitted conservation law arises from multipliers $\lambda^\nu(x, u^{(l)})$ such that

$$\lambda^\nu(x, u^{(l)})\Delta_\nu(x, u^{(n)}) = D_i P_i(x, u^{(r)}) \tag{22}$$

holds identically, where the summation convention is used whenever appropriate. Through this approach, the determining of conservation laws for a given PDE system (19) reduces to finding sets of multipliers.

By direct calculation, one can show that the Euler operators annihilate any divergence expression $D_i P_i(x, u^{(r)})$. Thus, the following identities hold for arbitrary function u:

$$E_{u^j}(D_i P_i(x, u^{(r)})) = 0, \quad j = 1, \ldots, q. \tag{23}$$

The converse also holds. Specifically, the only scalar expressions annihilated by Euler operators are divergence expressions. In continuation, the following theorem is applied which connecting multipliers and conservation laws. A set of multipliers $\{\lambda^\nu(x, u^{(l)})\}_{\nu=1}^N$ yields a conservation law for the PDE system (19) if and only if the set of identities

$$E_{u^j}(\lambda^{\nu}(x,u^{(l)})\Delta_{\nu}(x,u^{(n)}))=0, \quad j=1,\ldots,q. \tag{24}$$

holds identically. See[1] for more details.

The set of equations (24) yields the set of linear determining equations to find all sets of conservation law multipliers of the PDE system (19) by considering multipliers of all orders.

In this section, we construct conservation laws for the system of equations (1). Consider the multipliers of the form

$$\begin{aligned}&\lambda^1(x,t,u,u_x,u_{x^2},u_{x^3},v,v_x,v_{x^2},v_{x^3}),\\&\lambda^2(x,t,u,u_x,u_{x^2},u_{x^3},v,v_x,v_{x^2},v_{x^3})\end{aligned} \tag{25}$$

for the system of equations (1). The determining equations for multipliers is

$$E_u[\lambda^1(u_t-uu_x+v_x-\frac{1}{2}u_{x^2})]=0, \tag{26}$$

$$E_u[\lambda^2(v_t-(uv)_x+\frac{1}{2}v_{x^2})]=0,$$

$$E_v[\lambda^1(u_t-uu_x+v_x-\frac{1}{2}u_{x^2}]=0,$$

$$E_v[\lambda^2(v_t-(uv)_x+\frac{1}{2}v_{x^2})]=0$$

Therefore, after straightforward but tedious calculation, we conclude that

$$\begin{aligned}\lambda^1&=(2c_2u+c_3)v^2+((v_x+\frac{1}{3}u_{x^2}+\frac{2}{3}u^3)c_2\\&+c_3u^2+c_1t+c_5+2c_4u)v\\&+(\frac{1}{6}v_{x^3}+\frac{1}{3}u_xv_x+v_xu^2+\frac{2}{3}uv_{xx})c_2\\&+c_4v_x+c_3(\frac{1}{3}v_{xx}+uv_x)+c_6,\end{aligned} \tag{27}$$

$$\begin{aligned}\lambda^2&=-\frac{1}{6}c_2u^4+\frac{1}{3}c_3u^3+((2v-u_x)c_2+c_4)u^2\\&+(\frac{2}{3}c_2u_{xx}+c_3(2v-u_x)\\&+c_1t+c_5)u+(\frac{1}{3}v_{x^2}-\frac{1}{6}u_{x^3}+v^2-u_xv+\frac{1}{2}u_x{}^2)c_2\\&+c_1x+c_4(2v-u_x)\\&+\frac{1}{3}c_3u_{x^2}+c_7\end{aligned} \tag{28}$$

where $c_i, i=1,\cdots,7$ are constants. In continuation, we apply the first homotopy formula which is due to bluman and Anco to construct conservation laws of the system (1). It is described in detail in[1].

In the following, conserved vectors are represented by two components T_1 and T_2 which are conserved density and flux respectively. So, we obtain the following conserved densities and fluxes.

Case 1:

$$\lambda^1=tv, \lambda^2=tu+x,$$

$$T_1=tuv+xv,$$

$$T_2=\frac{1}{2}v-xuv+\frac{1}{2}t(u_xv-uv_x)-\frac{1}{2}xv_x-tu^2v-\frac{1}{2}tv^2$$

Case 2:

$$\begin{aligned}\lambda^1&=2uv^2+vv_x+\frac{1}{3}vu_{xx}+\frac{2}{3}vu^3+\frac{1}{6}v_{xxx}+\frac{1}{3}u_xv_x\\&+v_xu^2+\frac{2}{3}uv_{xx},\end{aligned}$$

$$\begin{aligned}\lambda^2&=\frac{1}{6}u^4+2vu^2-u^2u_x+\frac{2}{3}uu_{xx}+\frac{1}{3}v_{xx}-\frac{1}{6}u_{xxx}\\&+v^2-u_xv+\frac{1}{2}u_x^2,\end{aligned}$$

$$\begin{aligned}T_1&=\frac{1}{3}v^3+v^2u^2+\frac{1}{6}u^4v+\frac{1}{4}u^2(uv_x-vu_x)\\&-\frac{1}{3}u_xv^2+\frac{1}{6}u_x^2v+\frac{1}{3}uvv_x\\&+\frac{1}{9}uu_xv_x+\frac{1}{3}uvu_{xx}+\frac{2}{9}u^2v_{xx}+\frac{1}{6}vv_{xx}\\&-\frac{1}{12}vu_{xxx}+\frac{1}{12}uv_{xxx},\end{aligned}$$

$$\begin{aligned}T_2&=-uv^3-\frac{1}{6}u^5v-\frac{4}{3}v^2u^3+\frac{1}{4}vu^2u_t\\&-\frac{1}{12}v_xu^4+uv^2u_x+\frac{1}{3}u_tv^2\end{aligned}$$

$$-\frac{1}{4}u^3v_t - u^2vv_x - \frac{1}{3}uvv_t - \frac{1}{2}uv_x{}^2$$
$$-\frac{1}{2}v^2v_x + \frac{1}{6}v_xv_t - \frac{1}{6}u^2u_xv_x$$

$$+\frac{2}{9}uv_xu_t - \frac{1}{3}vuu_{tx} + \frac{1}{6}uv_xu_{xx} - \frac{1}{6}uu_xv_{xx}$$
$$+\frac{1}{6}vu_xv_x + \frac{1}{3}uu_xv_t$$

$$-\frac{1}{2}uu_x{}^2v + \frac{1}{3}vu^3u_x - \frac{1}{6}vv_{tx} - \frac{2}{9}u^2v_{tx}$$
$$+\frac{1}{12}v_{tx}u_x - \frac{1}{12}u_{tx}v_x$$

$$-\frac{1}{12}v_tu_{xx} + \frac{1}{12}u_tv_{xx} + \frac{1}{12}vu_{txx} - \frac{1}{12}uv_{txx}$$
$$+\frac{1}{12}v_{xx}u_{xx} - \frac{1}{12}v_xu_x{}^2$$

$$-\frac{1}{6}v_xv_{xx} + \frac{1}{6}vu_xu_{xx}$$

Case 3:

$$\lambda^1 = v^2 + vu^2 + uv_x + \frac{1}{3}v_{xx}, \lambda^2$$
$$= \frac{1}{3}u^3 + 2uv - uu_x + \frac{1}{3}u_{xx},$$

$$T_1 = \frac{1}{3}vu^3 + uv^2 - \frac{1}{3}vuu_x + \frac{1}{3}v_xu^2$$
$$+\frac{1}{6}vu_{xx} + \frac{1}{6}uv_{xx},$$

$$T_2 = -\frac{1}{3}v^3 - \frac{3}{2}v^2u^2 - \frac{1}{3}u^4v + \frac{1}{3}uvu_t - \frac{1}{6}u_x^2v$$
$$-\frac{1}{6}v_x^2 + \frac{1}{2}vu^2u_x$$

$$-uvv_x - \frac{1}{3}u^2v_t - \frac{1}{6}v_xu^3 + \frac{1}{2}u_xv^2 + \frac{1}{6}v_xu_t$$
$$+\frac{1}{6}u_xv_t + \frac{1}{6}uu_xv_x$$

$$-\frac{1}{6}vu_{tx} - \frac{1}{6}uv_{tx}$$

Case 4:

$$\lambda^1 = 2uv + v_x, \lambda^2 = u^2 + 2v - u_x,$$

$$T_1 = vu^2 + v^2 - \frac{1}{2}vu_x + \frac{1}{2}uv_x,$$

$$T_2 = -2uv^2 - vu^3 + uvu_x - vv_x$$
$$+\frac{1}{2}u_xv_x - \frac{1}{2}v_xu^2$$

$$-\frac{1}{2}uv_t + \frac{1}{2}vu_t$$

Case 5:

$$\lambda^1 = v, \lambda^2 = u,$$

$$T_1 = uv,$$

$$T_2 = -vu^2 - \frac{1}{2}v^2 + \frac{1}{2}u_xv - \frac{1}{2}uv_x$$

Case 6:

$$\lambda^1 = 1, \lambda^2 = 0,$$

$$T_1 = u, T_2 = -v - \frac{1}{2}u^2 + \frac{1}{2}u_x$$

Case 7:

$$\lambda^1 = 0, \lambda^2 = 1,$$

$$T_1 = v, T_2 = -uv - \frac{1}{2}v_x$$

4. Hamiltonian Symmetries

The correspondence between Hamiltonian symmetry groups and conservation laws for systems of evolution equations in Hamiltonian form is known as Nother's theorem, after the prototype [8]. This relationship has been analyzed extensively by Olver[9], Wilson[12]], Gel'fand and Dikii[6]. As we remarked in the last section, Any conservation law of a system of evolution equations takes the form

$$\mathrm{D}_t \mathrm{T} + \mathrm{Div}\mathrm{X} = 0,$$

in which Div denotes spatial divergence. Note that if $\mathrm{T}(x, t, u^{(n)})$ is any such differential function, and u is a solution to the evolutionary system $u_t = K[u]$, then

$$\mathrm{D}_t = \partial_t \mathrm{T} + \mathrm{pr}\mathbf{v}_K(\mathrm{T}). \tag{29}$$

where $\partial_t = \partial / \partial t$ denotes the partial t-derivative. Thus T is the density for a conservation law of the system if and only if its associated functional T satisfies

$$\partial \mathrm{T} / \partial t + \mathrm{pr}\mathbf{v}_K(\mathrm{T}) = 0. \tag{30}$$

In the case our system is of Hamiltonian form, the following proposition[9] is used. Let D be a Hamiltonian operator with poisson bracket (4). To each functional $\mathrm{H} = \int H dx$, there is an evolutionary vector field $\check{\mathbf{v}}_{\mathrm{H}}$, called the Hamiltonain vector field associated with H, which satisfies

$$\mathrm{pr}\check{\mathbf{v}}_{\mathrm{H}}(\mathrm{P}) = \{\mathrm{P}, \mathrm{H}\} \tag{31}$$

for all functionals P. Indeed, $\check{\mathbf{v}}_{\mathrm{H}}$ has characteristic $\mathrm{D}\delta\mathrm{H} = \mathrm{DE}(H)$. Hence, the bracket relation (31) immediately leads to the Noether relation between Hamiltonian symmetries and conservation laws.

So, for the system of equations WBK, generalized symmetries which are Hamiltonian can be deduced from conserved densities by using the Hamiltonian operators. The system of equations Whitham-Broer-Kaup admits Hamiltonian symmetries with the following characteristics, in the case of the Hamiltonian operator D_0,

$$Q_1^u = u_t, Q_1^v = v_t, \quad Q_2^u = u_x, Q_2^v = v_x$$
$$Q_3^u = tu_x + 1, Q_3^v = tv_x,$$

$$Q_4^u = (-u_x + 2v)u_x - uu_{xx} + \frac{1}{3}u_{xxx} + 2uv_x,$$
$$Q_4^v = v_x u_x + u^2 v_x + uv_{xx} + \frac{1}{3}v_{xxx},$$

$$Q_5^u = \frac{5}{3}u_x u_{xx} + \frac{1}{3}{u_x}^2 - u_{xx}u^2 - \frac{2}{3}vu_{xx} + \frac{1}{3}uu_{xx}$$
$$+ \frac{2}{3}uu_{xxx} - \frac{1}{6}u_{xxxx} + 2u^2 v_x$$
$$- \frac{2}{3}v_x u_x + \frac{1}{3}v_{xxx},$$

$$Q_5^v = 2u_x v^2 + v_{xx}u_x - \frac{1}{3}v_x u_x + \frac{2}{3}v_x u_{xx} + \frac{1}{3}vu_{xxx}$$
$$+ \frac{2}{3}v_x u^3 + \frac{2}{3}{v_x}^2 + v_{xx}u^2$$

$$+ \frac{2}{3}vv_{xx} - \frac{1}{3}uv_{xx} + \frac{2}{3}uv_{xxx} + \frac{1}{6}v_{xxxx}.$$

Also, generalized symmetries corresponding to the Hamiltonian operators D_1 can be deduced from the conservation laws. Thus, the Hamiltonian symmetries relative to D_1 are

$$Q_1^u = u_t, Q_1^v = v_t,$$

$$Q_2^u = 2tu_t + xu_x + u, Q_2^v = 2tv_t$$
$$+ xv_x + 2v,$$

$$Q_3^u = 4uv_x + 4vu_x + u(-u_{xx} + 2v_x)$$
$$+ u^2 u_x + 2v - u_x + u_{xxx},$$

$$Q_3^v = 2uv_t + 4v_x u_x + 2uv_{x^2} + v_{xxx}$$
$$+ 4vv_x + u^2 v_x + 2v - u_x,$$

$$Q_4^u = 2u^2 v_x + \frac{2}{3}v_{xxx} - 2u{u_x}^2 + 4vuu_x - u_{xx}u^2$$
$$+ \frac{4}{3}uu_{xxx} + 2u^2 v_x + \frac{1}{3}u_x u^3$$

$$+ \frac{10}{3}u_x u_{xx} - 2v_x u_x - 2vu_{xx} - \frac{1}{3}u_{xxxx},$$

$$Q_4^v = u^3 v_x + v_{xx}u^2 + \frac{4}{3}uv_{xxx} + 2v_{xx}u_x + \frac{4}{3}v_x u_{xx}$$
$$+ v_{xx}u^2 + \frac{1}{3}v_{xxxx} - 2{u_x}^2 v$$

$$+4u_x v^2 - 2vuu_{xx} + \frac{2}{3}vu_{xxx}$$

$$+6vuv_x + \frac{1}{3}v_x u^3.$$

5. Conclusion

In this paper, the tri-Hamiltonian system of equations Whitham-Broer-Kaup is studied. conservation laws of the system by considering multipliers of order three are constructed. Moreover, generalized symmetries of the system of equations WBK which are Hamiltonian are obtained by using the Hamiltonian operators and conserved densities.

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