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Optimal Control of an N-Policy Two-Phase $M^X/E_k/1$ Queuing System with Server Startup Subject to the Server Breakdowns and Delayed Repair

V. Vasanta Kumar[1*], T. Srinivasa Rao[1], B. Srinivasa Kumar[1] and B. V. S. N. Hari Prasad[2]

[1]K. L. University, Vaddeswaram, Guntur (District) - 522502, Andhra Pradesh, India; vvkumar@kluniversity.in

[2]Dhanekula Institute of Technology, Vijayawada - 521139, Andhra Pradesh, India; tsr_2505@kluniversity.in, sk_bhavirisetty@kluniversity.in

Abstract

Background/Objective: To investigate the control policy of the two-phase $M^x/E_k/1$ queueing system with server startup time, N-policy and unreliable server, consisting of breakdown and delay periods. **Methods:** Steady state equations governing the queuing system are written and is solved using the probability generating functions to obtain the steady state probabilities when the server is in different states. **Results:** Expected queue length when the server is in different states is derived. Also the expected waiting time in the queue is obtained through heuristic approach. Optimal threshold of N that minimizes the average cost per unit time is derived. Sensitivity analysis is performed on the optimal threshold N^* based on changes in the system parameters and cost elements for the geometric arrival batch size distribution. **Conclusions/Applications:** This model can be obtained in the two-phase service systems like communication networks and production systems to analyze the system performance.

Keywords: Breakdowns, Cost Function, Delay Time, N-Policy, Repair Time, Two-Phase, Vacation

1. Introduction

In most queuing systems the server may be subjected to lengthy and unpredictable breakdowns while serving a customer. For instance, in manufacturing systems the machine may breakdown due to malfunction or job related problems. In these systems, server breakdown results in a period of server unavailability time until it is fixed. Therefore, it is necessary to see how the breakdowns affect the level of performance of the system.

Regarding the queuing systems with two phases of service Krishna and Lee[5] and Doshi[4] studied the distributed systems where all customers waiting in the queue receive batch service in the first phase of service followed by individual service in the second phase. Selvam and Sivasankaran[7] introduced the two phase queuing system with server vacations. Kim and Chae[6] investigated the two phase system with N-policy. The server startups correspond to the preparatory work of the server before starting the service. In some actual situations, the server often requires a startup time before starting each service period. Baker[1] first proposed the N-policy M/M/1 queuing system with exponential startup time. Several authors have investigated queuing models with server breakdowns and vacations in different frameworks in recent past. Wang[15] for the first time proposed the Markovian queuing system under the N-policy with server breakdowns. Vasanta Kumar and Chandan[8] presented the optimal operating policy for the two-phase $M^x/E_k/1$ queueing system under N-policy. Vasanta Kumar et al.[9,10] presented the optimal control of $M^x/M/1$ and $M/E_k/1$ gated queuing systems with server startup and break downs. Vasanta Kumar et al.[11,12] presented the optimal operating policy for a two-phase $M^x/M/1$ and $M^x/E_k/1$ queueing systems under N-policy with server breakdowns and without gating. Choudhury and Tadj[2] considered an M/G/1 model with an additional

*Author for correspondence

phase of optional service with the assumption that the server is subject to breakdowns and delayed repair. Later Choudury et al.[3] investigated such a type of model, where the concept of N-policy is also introduced along with a delayed repair for batch arrival queueing system. In the two-phase $M^x/M/1$ queueing system without gating analyzed by Vasanta Kumar et al.[11] breakdowns are considered in second phase of service, without taking into consideration the concept of delayed repair.

Hence the present paper aims at the study of economic behavior of an N-policy $M^x/E_k/1$ queue in which service is in two-phases and the server is typically subjected to unpredictable breakdowns in both phases of service and delayed repair.

The further discussions of the paper are organized as follows:

In section 2 the assumptions of the model are presented. Section 3 deals with the steady state results and expected number of customers in the system when the server is in different states. Section 4 deals with some other system performance measures. Optimal control policy is presented in section 5. Sensitivity analysis with numerical illustrations is presented in section 6. Conclusions are presented in section 7.

2. Model Description and Assumptions

We consider an $M^x/E_k/1$ queuing system with server startup, two-phases of service, unreliable server, and delay in repair due to non-availability of the service facility operating under N-policy.

2.1 Assumptions of the Model

- The arrival process is a compound Poisson process (with rate λ) of independent and identically distributed random batches of customers, where each batch size X, has a probability density function $\{a_n: a_n = P(X=n), n \geq 1\}$. Batches are admitted to service on a first come first served basis.
- The service is in two phases. The first phase of service is batch service to all customers waiting in the queue. On completion of batch service the server proceeds to second phase to serve all customers in the batch individually. Batch service time is assumed to be exponentially distributed with mean $1/\beta$ and is independent of batch size. Individual service is in k exponentially distributed phases with mean $1/k\mu$. On completion of individual service, the server returns to the batch queue to serve the customers who have arrived. If customers are waiting, the server starts the batch service followed by individual service to each customer in the batch. If no customer is waiting, the server takes a vacation.
- Whenever, the system becomes empty, the server is turned off. As soon as the total number of arrivals in the queue reaches to a predetermined threshold N the server is turned on and is temporarily unavailable for the waiting customers to restart service. It needs a startup time which follows an exponential distribution with mean $1/\theta$. As soon as the server finishes startup, it starts serving the first phase of waiting customers.
- The customers who arrive during the pre-service and batch service are also allowed to enter the same batch which is in service.
- The server is subject to breakdowns at any time with Poisson breakdown rates α_1 for the first phase of service and α_2 for the second phase of service where it is working. Whenever, the server fails, it is sent for repair during which the server stops providing service and waits for the repair to get started. The waiting time for repair is defined as delay time and is assumed to be exponentially distributed with mean $1/\delta$. Repair time in any phase of service is assumed to be exponentially distributed with mean $1/\gamma$.
- In case the server breaks down while serving customers, it is sent for repair and that particular batch of customers or the customer who is just being served should wait for the server to come back to complete the remaining service. Immediately after the server is repaired, it starts to serve and the service time is cumulative. A customer who arrives and finds the server busy or broken down must wait in the queue until a server is available. Customers continue to arrive during the delay and repair periods of the broken server.

3. Steady State Results

In steady state the following notations are used.

$P_{0,i,0}$ = The probability that there are i service phases in the system and the server is on vacation, I = 0, k, 2k, 3k,...(N-1)k

$P_{1,i,0}$ = The probability that there are i service phases in the in the batch queue and the server is doing pre-service (startup work), where i = Nk, (N+1)k, (N+2)k,…

$P_{2,i,0}$ = The probability that there are i service phases in the batch which is in the batch service, i = k, 2k, 3k…

$P_{3,i,0}$ = The probability that there are i service phases in the batch which is in batch service, but the server is found to be broken down and waiting for repair, i = k, 2k, 3k,…

$P_{4,i,0}$ = The probability that there are i service phases in the batch which is in batch service, but the server is under repair, i = k, 2k, 3k,…

$P_{5,i,j}$ = The probability that there are i service phases in the batch queue and j service phases in the individual queue while the server is in individual service, i = 0, k, 2k, 3k,… and j = 1, 2, 3,…

$P_{6,i,j}$ = The probability that there are i service phases in the batch queue and j service phases in the individual queue when the server is in individual service , but found to be broken down and waiting for repair, i = 0, k, 2k, 3k,…, and j = 1,2,3….

$P_{7,i,j}$ = The probability that there are i service phases in the batch queue and j service phases in the individual queue when the server is in individual service, but the server is under repair, i = 0, k, 2k, 3k…, and j = 1,2,3….

The steady state equations satisfied by the system size probabilities are as follows:

$$\lambda P_{0,0,0} = k\mu P_{5,0,1}. \tag{1}$$

$$\lambda P_{0,i,0} = \lambda \sum_{l=k}^{i} c_l p_{0,i-l,0}, k \le i \le (N-1)k. \tag{2}$$

$$(\lambda+\theta)\, P_{1,Nk,0} = \lambda \sum_{l=k}^{Nk} c_l p_{0,Nk-l,0}. \tag{3}$$

$$(\lambda+\theta)P_{1,i,0} = \lambda \sum_{l=k}^{i-Nk} c_l p_{1,i-l,0} + \lambda \sum_{l=i-(N-1)k}^{i} c_l p_{0,i-l,0}, \quad i \ge (N+1)k. \tag{4}$$

$$(\lambda+\beta+\alpha_1)\, P_{2,i,0} = \lambda \sum_{l=k}^{i} c_l p_{2,i-l,0} + k\mu\, P_{5,i,1} + \gamma\, P_{4,i,0}, \quad k \le i \le (N-1)k. \tag{5}$$

$$(\lambda+\beta+\alpha_1)P_{2,i,0} = \lambda \sum_{l=k}^{i} c_l p_{2,i-l,0} + k\mu\, P_{5,i,1} + \gamma\, P_{4,i,0} + \theta\, p_{1,i,0}, \; i \ge Nk. \tag{6}$$

$$(\lambda+\delta)\, P_{3,k,0} = \alpha_1\, P_{2,k,0}. \tag{7}$$

$$(\lambda+\delta)\, P_{3,i,0} = \alpha_1\, P_{2,i,0} + \lambda \sum_{l=k}^{i} c_l p_{3,i-l,0}, \; i \ge 2k \tag{8}$$

$$(\lambda+\gamma)\, P_{4,k,0} = \delta\, P_{3,k,0}. \tag{9}$$

$$(\lambda+\gamma)\, P_{4,i,0} = \delta\, P_{3,i,0} + \lambda \sum_{l=k}^{i} c_l P_{4,i-l,0}, \; i \ge k. \tag{10}$$

$$(\lambda+\alpha_2+k\mu)\, P_{5,0,j} = k\mu\, P_{5,0,j+1} + \beta\, P_{2,j,0} + \gamma\, P_{7,0,j}, \; j \ge 1. \tag{11}$$

$$(\lambda+\alpha_2+k\mu)\, P_{5,i,j} = k\mu\, P_{5,i,j+1} + \lambda \sum_{l=k}^{i} c_l\, P_{5,i-l,j} + \gamma\, P_{7,i,j}, \; i \ge k, j \ge 1. \tag{12}$$

$$(\lambda+\delta)\, P_{6,0,j} = \alpha_2\, P_{5,0,j}, \; j \ge 1. \tag{13}$$

$$(\lambda+\delta)\, P_{6,i,j} = \alpha_2 P_{5,i,j} + \lambda \sum_{l=k}^{i} c_l\, P_{6,i-l,j}, \; i \ge k, \; j \ge 1. \tag{14}$$

$$(\lambda+\gamma)\, P_{7,0,j} = \delta\, P_{6,0,j}, \; j \ge 1. \tag{15}$$

$$(\lambda+\gamma)\, P_{7,i,j} = \delta\, P_{6,i,j} + \lambda \sum_{l=k}^{i} c_l\, P_{7,i-l,j}, \; i \ge k, \; j \ge 1. \tag{16}$$

The following generating functions are used to solve the equations (1) to (16)

$$G_0(z) = \sum_{i=0}^{(N-1)k} P_{0,i,0} z^i, G_1(z) = \sum_{i=Nk}^{\infty} P_{1,i,0}\, z^i, G_2(z) = \sum_{i=k}^{\infty} P_{2,i,0}\, z^i,$$

$$G_3(z) = \sum_{i=k}^{\infty} P_{3,i,0}\, z^i, \; G_4(z) = \sum_{i=k}^{\infty} P_{4,i,0}\, z^i, \; G_5(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_{5,i,j} z^i\, y^j,$$

$$G_6(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_{6,i,j}\, z^i y^j, \quad G_7(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_{7,i,j}\, z^i y^j, R_j(z) = \sum_{i=0}^{\infty} p_{5,i,j} z^i,$$

$$S_j(z) = \sum_{i=0}^{\infty} p_{6,i,j} z^i, T_j(z) = \sum_{i=0}^{\infty} p_{7,i,j} z^i, \; |z| < 1 \text{ and } |y| < 1.$$

Let $C(z) = \sum_{i=k}^{\infty} c_i z^i$ be the probability generating function of the number of service phases in the arrival batch. $C^1(z)$ and $C^{11}(z)$ represent the first and second order derivatives of $C(z)$. It is evident that $E(C) = C^l(1)$ and $E(C(C-1)) = C^{ll}(1)$.

Let $A(z) = \sum_{i=1}^{\infty} a_i z^i$ be the probability generating function of the arrival batch size random variable X and $A^l(z)$, $A^{ll}(z)$ represent the first and second order derivatives of $A(z)$ respectively.

It can be shown that $C^l(1) = k A^l(1)$ and $C^{11}(1) = k^2 A^{ll}(1)+k(k-1)A^l(1)$

Solving equations (1) to (16) the following generating functions are obtained:

$$G_0(z) = P_{0,0,0}\, y_N(z), \text{ where } y_N(z) = \sum_{i=0}^{(N-1)k} y_i z^i,\ y_i = \sum_{l=k}^{i} c_l y_{i-l},\ y_0 = 1, \tag{17}$$

$$[\lambda(1-C(z))+\theta]G_1(z) = \lambda P_{0,0,0} + \lambda (C(z)-1)G_0(z), \tag{18}$$

$$[\lambda(1-C(z))+\beta+\alpha_1]G_2(z) = k\mu R_1(z)+\gamma G_4(z)+\theta G_1(z)-\lambda P_{0,0,0}, \tag{19}$$

$$[\lambda(1-C(z))+\delta]G_3(z) = \alpha_1 G_2(z), \tag{20}$$

$$[\lambda(1-C(z))+\gamma]G_4(z) = \delta G_3(z), \tag{21}$$

$$\begin{aligned}[\lambda y(1-C(z))+\alpha_2 y+k\mu(y-1)]\,G_5(z,y) \\ = -k\mu y R_1(z)+\gamma y G_7(z,y)+\beta y G_2(y),\end{aligned} \tag{22}$$

$$[\lambda(1-C(z))+\delta]G_6(z,y) = \alpha_2 G_5(z,y), \tag{23}$$

$$[\lambda(1-C(z))+\gamma]G_7(z,y) = \delta G_6(z,y). \tag{24}$$

The total probability generating function $G(z,y)$ is given by

$$\begin{aligned}G(z,y) = G_0(z)+G_1(z)+G_2(z)+G_3(z)+G_4(z) \\ +G_5(z,y)+G_6(z,y)+G_7(z,y).\end{aligned} \tag{25}$$

The normalizing condition is

$$\begin{aligned}G(1,1) = G_0(1)+G_1(1)+G_2(1)+G_3(1)+G_4(1) \\ +G_5(1,1)+G_6(1,1)+G_7(1,1)=1.\end{aligned} \tag{26}$$

This condition yields, $R_1(1) = \lambda C^1(1)/(k^2\mu)$.

Under steady state conditions, let P_v, P_s, P_b, P_{bb}, P_{db}, P_i, P_{bi}, and P_{di}, be the probabilities that the server is in vacation, in startup, in batch service, waiting for repair during batch service, under repair during batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively. Then

$$P_v = G_0(1) = y_N(1)P_{0,0,0}, \tag{27}$$

$$P_s = G_1(1) = (\lambda P_{0,0,0}/\theta), \tag{28}$$

$$P_b = G_2(1) = (\lambda C^1(1)/k\beta), \tag{29}$$

$$P_{bb} = G_3(1) = (\alpha_1/\delta)(\lambda C^1(1)/k\beta), \tag{30}$$

$$P_{db} = G_4(1) = (\alpha_1/\gamma)(\lambda C^1(1)/k\beta), \tag{31}$$

$$P_i = G_5(1,1) = (\lambda C^1(1)/k\mu), \tag{32}$$

$$P_{bi} = G_6(1,1) = (\alpha_2/\delta)(\lambda C^1(1)/k\mu), \tag{33}$$

$$P_{di} = G_7(1,1) = (\alpha_2/\gamma)(\lambda C^1(1)/k\mu), \tag{34}$$

Probability that the server is neither doing batch service nor individual service is given by

$$G_0(1)+G_1(1) = 1-\left(\frac{\lambda A^1(1)}{k\beta}\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)+\frac{\lambda A^1(1)}{k\mu}\left(1+\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)\right)$$

This gives

$$P_{0,0,0} = (1-\rho)\frac{\theta}{(\lambda+y_N(1))\theta_1}, \tag{35}$$

$$\text{where}\quad \rho = \frac{\lambda C^1(1)}{k\beta}\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)+\frac{\lambda C^1(1)}{k\mu}\left(1+\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)$$

is the utilizing factor of the system.

From Equation (35) we have $\rho < 1$, which is the necessary and sufficient condition under which steady state solution exits.

3.1 Expected Number of Customers in Different States

Using the probability generating functions expected number of customers in the system at different states are derived in this section.

Let L_v, L_s, L_b, L_{bb}, L_{db}, L_i, L_{bi}, and L_{di}, be the expected number of customers in the system when the server is in vacation, in startup, in batch service, waiting for repair during batch service, under repair during batch service, in individual service, waiting for repair during individual service and under repair during individual service states respectively.

Differentiating the generating functions $G_q(z)$, q = 0,1,2,3,4,5,6,7 given in equations (17) to (24) and substituting z =1 we get

$$L_v = G_0{}^1(1) = y_N{}^1(1)P_{0,0,0}, \tag{36}$$

$$L_s = G_1{}^1(1) = [\lambda\, C^1(1)\left(\lambda + \theta\, y_N(1)\right)/\theta^2]\; P_{0,0,0}, \tag{37}$$

$$L_b = G_2{}^1(1) = \lambda\, C^1(1)/\beta, \tag{38}$$

$$L_{bb} = G_3{}^1(1) = \left(\frac{\lambda\alpha_1 C^1(1)}{\beta\delta}\right)\left(1 + \frac{\lambda\, C^1(1)}{k\,\delta}\right), \tag{39}$$

$$L_{db} = \sum_{i=1}^{\infty} i\, P_{4,i,0} = G_4{}'(1) = \frac{\lambda\,\alpha_1 C^1(1)}{\gamma\,\beta}\left(1 + \frac{\lambda\, C^1(1)}{k\,\delta} + \frac{\lambda\, C^1(1)}{k\,\gamma\,\delta}\right), \tag{40}$$

$$\begin{aligned} L_i &= G_5{}^1(1,1) \\ &= \frac{\lambda\, C^1(1)}{k\mu}\left[\frac{1}{(1-\rho_2)} + \frac{k\,\rho_1}{(1-\rho_2)} + \frac{\lambda\, C^1(1)}{(1-\rho_2)}\left(\frac{\lambda + \theta\, y_N(1)}{\theta^2}\right)P_{0,0,0}\right. \\ &\quad + \frac{\lambda\, C^1(1)\,\rho_2}{(1-\rho_2)}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{C^{11}(1)}{2\, C^1(1)(1-\rho_2)}\left(\frac{\lambda + \theta\, y_N(1)}{\theta}\right)P_{0,0,0} \\ &\quad + \frac{y_N{}^1(1)}{(1-\rho_2)}P_{0,0,0} + \frac{(\lambda C^1(1))^2\alpha_1}{k\,\beta\,\gamma\,\delta\,(1-\rho_2)}\left(1 + \frac{\delta}{\gamma} + \frac{\gamma}{\delta}\right) + \frac{C^{11}(1)\,(\rho_1 + \rho_2)}{2\, C^1(1)(1-\rho_2)} \\ &\quad \left. - \frac{(\lambda C^1(1))^2}{k\,\mu\,(1-\rho_2)}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) - \frac{(\lambda C^1(1))^2\alpha_2}{k\,\mu\,\gamma\,\delta\,(1-\rho_2)}\right], \end{aligned}$$

$$L_{bi} = G_6{}^1(1,1) = \frac{(\lambda C^1(1))^2\alpha_2}{k\,\mu\,\delta^2} + \frac{\alpha_2}{\delta}G_5{}^1(1,1), \tag{41}$$

$$L_{di} = G_7{}^1(1,1) = \frac{(\lambda C^1(1))^2\alpha_2}{k\,\mu\,\gamma}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{\alpha_2}{\gamma}G_5{}^1(1,1), \tag{42}$$

where

$\rho_1 = \dfrac{\lambda C^1(1)}{k\,\beta}\left(1 + \dfrac{\alpha_1}{\gamma} + \dfrac{\alpha_1}{\delta}\right)$, $\rho_2 = \dfrac{\lambda C^1(1)}{k\,\mu}\left(1 + \dfrac{\alpha_2}{\gamma} + \dfrac{\alpha_2}{\delta}\right)$ and $G_q{}^1(.)$ denotes the first order derivative of $G_q(.)$.

The expected number of service phases in the system is given by

$$\begin{aligned} L(P) &= L_v + L_s + L_b + L_{bb} + L_{db} + L_i + L_{bi} + L_{di} \\ &= \frac{\rho_2}{(1-\rho_2)} + \frac{k\,\rho_1}{(1-\rho_2)} + \frac{y_N^1(1)}{(1-\rho_2)}P_{0,0,0} + \frac{\lambda\, C^1(1)}{(1-\rho_2)}\left(\frac{\lambda + \theta\, y_N(1)}{\theta^2}\right)P_{0,0,0} \\ &\quad + \frac{(\lambda\, C^1(1))^2\,\alpha_1}{\beta\,\gamma\,\delta\, k(1-\rho_2)}\left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) + \frac{(\lambda\, C^1(1))^2\alpha_2}{k\,\mu\,\gamma\,\delta}\left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) \\ &\quad + \frac{\lambda\, C^1(1)\rho_2}{(1-\rho_2)}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right)\left(\rho_2 - \frac{\lambda C^1(1)}{k\mu}\right) - \frac{(\lambda C^1(1))^2\alpha_2\,\rho_2}{k\,\mu\,\gamma\,\delta(1-\rho_2)} \\ &\quad + \frac{C^{11}(1)\rho_2}{2C^1(1)\,(1-\rho_2)}\left(\frac{\lambda + \theta\, y_N(1)}{\theta}\right)P_{0,0,0} + \frac{C^{11}(1)\,\rho_2\rho}{2\, C^1(1)\,(1-\rho_2)} \end{aligned} \tag{43}$$

Then the expected number of customers in the system is given by

$$\begin{aligned} L(N) &= \frac{1}{k}\left[L(P) - \left(\frac{k+1}{2}\right)\left(\frac{\lambda}{k\,\mu}\left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta}\right) + \frac{\lambda}{k\,\beta}\left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta}\right)\right)C^1(1)\right] \\ &\quad + \left(\frac{\lambda}{k\mu}\left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta}\right) + \frac{\lambda}{k\beta}\left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta}\right)\right)C^1(1) \end{aligned} \tag{44}$$

Expressing this in terms of $A^1(1)$

$$\begin{aligned} L(N) &= \frac{1}{k}\left[L(P) - \left(\frac{k+1}{2}\right)\left(\frac{\lambda}{\mu}\left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta}\right) + \frac{\lambda}{\beta}\left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta}\right)\right)A^1(1)\right] \\ &\quad + \left(\frac{\lambda}{\mu}\left(1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta}\right) + \frac{\lambda}{\beta}\left(1 + \frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta}\right)\right)A^1(1), \end{aligned} \tag{45}$$

where

$$\begin{aligned} L(P) &= \frac{\rho_2}{(1-\rho_2)} + \frac{k\,\rho_1}{(1-\rho_2)} + \frac{y_N{}^1(1)}{(1-\rho_2)}P_{0,0,0} + \frac{\lambda\, k\, A^1(1)}{(1-\rho_2)}\left(\frac{\lambda + \theta\, y_N(1)}{\theta^2}\right)P_{0,0,0} \\ &\quad - \frac{\left(\lambda\, A^1(1)\right)^2 k\alpha_2\rho_2}{\mu\gamma\delta(1-\rho_2)} + \frac{\left(\lambda\, A^1(1)\right)^2 k\,\alpha_1}{\beta\gamma\delta\,(1-\rho_2)}\left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) \\ &\quad + \frac{(\lambda\, A^1(1))^2\, k\,\alpha_2}{\mu\gamma\delta}\left(1 + \frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) + \frac{(kA^{11}(1) + (k-1)A^1(1))\,\rho_2\,\rho}{2\, A^1(1)\,(1-\rho_2)} \\ &\quad + \frac{\lambda k\, A^1(1)\rho_2}{(1-\rho_2)}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right)\left(\rho_2 - \frac{\lambda\, A^1(1)}{\mu}\right) + \frac{(kA^{11}(1) + (k-1)A^1(1))\,\rho_2}{2\, A^1(1)\,(1-\rho_2)} \\ &\quad \left(\frac{\lambda + \theta\, y_N(1)}{\theta}\right)P_{0,0,0}, \end{aligned} \tag{46}$$

$$\rho_1 = \frac{\lambda A^1(1)}{\beta}\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right),\ \rho_2 = \frac{\lambda\ A^1(1)}{\mu}\left(1\ +\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)$$
$$\text{and } \rho = \rho_1 + \rho_2$$

4. Some Other System Performance Measures

In this section, expected length of vacation period, startup period, batch service period, delay period during batch service, waiting period for repair during batch service, individual service period, delay period during individual service and waiting period for repair during individual service are presented.

The expected length of a busy cycle E_c is given by

$$E_c = E_v + E_s + E_b + E_{bb} + E_{db} + E_i + E_{bi} + E_{di}. \quad (47)$$

The long run fractions of time the server is in different states are respectively,

$$E_v / E_c = P_v = y_N(1)\ P_{0,0,0}, \quad (48)$$

$$E_s / E_c = P_s = (\lambda/\theta)\ P_{0,0,0}, \quad (49)$$

$$E_b / E_c = P_b = \lambda\ A^1(1)/\beta\ , \quad (50)$$

$$E_{bb} / E_c = P_{bb} = (\alpha_1/\delta)\ \left(\lambda\ A^1(1)/\beta\right)\ , \quad (51)$$

$$E_{db} / E_c = P_{db} = (\alpha_1/\gamma)\ \left(\lambda\ A^1(1)/\beta\right), \quad (52)$$

$$E_i / E_c = P_i = \lambda\ A^1(1)/\mu\ , \quad (53)$$

$$E_{bi} / E_c = P_{bi} = (\alpha_2/\delta)\ \left(\lambda A^1(1)/\mu\right), \quad \text{and} \quad (54)$$

$$E_{di} / E_c = P_{di} = (\alpha_2/\gamma)\ \left(\lambda A^1(1)/\mu\right). \quad (55)$$

Expected length of vacation period is given by

$$E_v = y_N(1)\ /\ (\lambda\ A^1(1)). \quad (56)$$

Substituting this in equation (48)

$$E_c = 1\ /(\lambda\ A^1(1)\ P_{0,0,0}). \quad (57)$$

4.1 Heuristic Approach to Waiting Time in the Queue

Let W_q be the waiting time of the test customer until his individual service. An arbitrary customer waits different time amounts according to the state of his arriving epoch. First, we divide the regeneration cycle into eight parts of the idle period, the startup period, the first phase batch service period, waiting time for repair and repair period due to breakdown in first phase, the second phase individual service period, waiting time for repair and repair period due to breakdown in second phase, and the repair period with respective probabilities

$$y_N(1)P_{0,0,0}, \frac{\lambda\ P_{0,0,0}}{\theta}, \frac{\lambda\ A'(1)}{\beta}, \frac{\alpha_1}{\delta}\left(\frac{\lambda\ A'(1)}{\beta}\right), \frac{\alpha_1}{\gamma}\left(\frac{\lambda\ A'(1)}{\beta}\right),$$
$$\frac{\lambda\ A'(1)}{\mu}, \frac{\alpha_2}{\delta}\left(\frac{\lambda\ A'(1)}{\mu}\right) \text{ and } \frac{\alpha_2}{\gamma}\left(\frac{\lambda\ A'(1)}{\mu}\right).$$

That is the system state that the arriving customer sees determines his waiting time. The test customer has to wait during the individual service times for those already waiting (except the ongoing individual service) in the system. In addition to it,

- If the server is idle, the customer has to wait the remaining idle period, startup period, the first phase batch service period.
- If the server is in the startup state, the customer has to wait the remaining startup period, the first phase batch service period.
- If the server is in the first phase, the customer has to wait the remaining time of the ongoing batch service.
- If the server is in the breakdown waiting state of batch service, the customer has to wait the remaining waiting period and repair period.
- If the server is in the repair state due to breakdown during batch service, the customer has to wait the remaining repair period.
- If the server is in the second phase, the customer has to wait the remaining time of the ongoing individual service plus the batch service.
- If the server is waiting for repair due to breakdown during individual service, the customer has to wait the remaining waiting period and the repair period, the first phase batch service period.

- If the server is in the repair state due to breakdown during individual service, the customer has to wait the remaining repair time period plus the first phase batch service period.

Thus,

$$E\left(W_q\right)=L(N-1)\frac{1}{\mu}+\left(\frac{N-1}{2\lambda A'(1)}+\frac{1}{\theta}+\frac{1}{\beta}\right)y_N(1)P_{0,0,0}+\left(\frac{1}{\theta}+\frac{1}{\beta}\right)\frac{\lambda P_{0,0,0}}{\theta}+\left(\frac{\lambda A'(1)}{\beta^2}\right)$$

$$+\left(\frac{1}{\delta}+\frac{1}{\gamma}+\frac{1}{\beta}\right)\left(\frac{\alpha_1}{\delta}\right)\left(\frac{\lambda A'(1)}{\beta}\right)+\left(\frac{1}{\gamma}+\frac{1}{\beta}\right)\left(\frac{\alpha_1}{\gamma}\right)\left(\frac{\lambda A'(1)}{\beta}\right)+\left(\frac{\lambda A'(1)}{\mu}\right)\left(\frac{1}{\mu}+\frac{1}{\beta}\right)$$

$$+\left(\frac{1}{\delta}+\frac{1}{\gamma}+\frac{1}{\beta}\right)\left(\frac{\alpha_2}{\delta}\right)\left(\frac{\lambda A'(1)}{\beta}\right)+\left(\frac{1}{\gamma}+\frac{1}{\beta}\right)\left(\frac{\alpha_2}{\gamma}\right)\left(\frac{\lambda A'(1)}{\mu}\right).$$

$$=\frac{1}{1-r}\left[\left(\frac{N-1}{2\lambda A'(1)}+\frac{1}{\theta}+\frac{1}{\beta}\right)y_N(1)P_{0,0,0}+\left(\frac{1}{\theta}+\frac{1}{\beta}\right)\left(\frac{\lambda P_{0,0,0}}{\theta}\right)+\left(\frac{\lambda A'(1)}{\beta^2}\right)\right.$$

$$+\left(\frac{1}{\delta}+\frac{1}{\gamma}+\frac{1}{\beta}\right)\left(\frac{\alpha_1}{\delta}+\frac{\alpha_2}{\gamma}\right)\left(\frac{\lambda A'(1)}{\beta}+\frac{\lambda A'(1)}{\mu}\right)+\left(\frac{\lambda A'(1)}{\mu}\right)\left(\frac{1}{\mu}+\frac{1}{\beta}\right)$$

$$\left.+\left(\frac{1}{\gamma}+\frac{1}{\delta}\right)\left(\frac{\alpha_1}{\delta}+\frac{\alpha_2}{\gamma}\right)\left(\frac{\lambda A'(1)}{\beta}+\frac{\lambda A'(1)}{\mu}\right)\right] \tag{58}$$

where $r=\left(\frac{\lambda A'(1)}{\mu}\right)-\left(\frac{\lambda A'(1)}{\beta}\right)$

4.2 Reliability Indices

In this section two reliability indices of the system viz.- the system availability and failure frequency under the steady state conditions are discussed. Let $A_v(T)$ be the system availability at time t, that is the probability that the server is working for a customer or in idle period or in startup period, or in batch service such that the steady state availability of the server will be

$$A_y = \underset{t\to\infty}{Lt} A_v(t)$$

The steady state availability of the server will be given by

$G_0(1)+G_1(1)+G_2(1)+G_5(1,1)$

$$=\frac{\left(\lambda+\theta\, y_N(1)\right)}{\theta}P_{0,0,0}+\frac{\lambda A'(1)}{\beta}+\frac{\lambda A'(1)}{\mu}$$

$$=1-\frac{\lambda A'(1)}{\beta}\left(\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)-\frac{\lambda A'(1)}{\mu}\left(\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)$$

The steady state failure frequency of the server is given by

$$M_f=\frac{\lambda\,\alpha_1 A'(1)}{\beta}+\frac{\lambda\,\alpha_2 A'(1)}{\mu}.$$

5. Optimal Cost Structure

In this section, the optimal value of N is determined that minimizes the long run average cost of two-phase $M^x/E_k/1$, N-policy queue with server break downs and delay in repair. To determine the optimal value of N the following linear cost structure is considered.

Let $C_A(N)$ be the average cost per unit of time, then

$$C_A(N)=C_h\, L(N)+C_0\left(\frac{E_b}{E_c}+\frac{E_i}{E_c}\right)+C_m\left(\frac{E_s}{E_c}\right)$$

$$+C_b\left(\frac{E_{bb}+E_{db}+E_{bi}+E_{di}}{E_c}\right)+C_s\left(\frac{1}{E_c}\right)-C_r\left(\frac{E_v}{E_c}\right) \tag{59}$$

where

$C_h \equiv$ Holding cost per unit of time spend for each customer present in the system,

$C_0 \equiv$ Cost per unit of time for keeping the server on and in operation,

$C_m \equiv$ Startup cost per unit time,

$C_s \equiv$ Setup cost per cycle,

$C_b \equiv$ Break down cost per unit of time for the unreliable server, and

$C_r \equiv$ Reward per unit of time as the server is doing secondary work during vacation.

From equations (50) to (55) it is observed that E_b/E_c, E_{bb}/E_c, E_{db}/E_c, E_i/E_c, E_{bi}/E_c, E_{di}/E_c, are not functions of the decision variable N.

Hence, for determination of the optimal operating N-policy, minimizing $C_A(N)$ in equation (59) is equivalent to minimizing.

$$T_A(N)=\frac{(1-\rho)\left[C_h\, Y_N^{\,1}(1)+(1-\rho_2)\left\{C_m\frac{\lambda}{\theta}+\lambda\, C_s-C_r Y_N(1)\right\}\right]}{(1-\rho_2)(y_N(1)+(\lambda/\theta))} \tag{60}$$

It is hard to prove that $T_A(N)$is convex. But a procedure that makes it possible to calculate the optimal threshold N* is presented below.

6. Result

Under the long run expected average cost criterion, the optimal threshold N^* for the model is the best value of 'k' given by,

$$N^* = \min.\left\{k \geq 1/\sum_{j=0}^{N}(k-j)y_j + \frac{k\lambda}{\theta} > \frac{\lambda(1-\rho_2)}{C_h}\left(\frac{C_m - C_r}{\theta} + C_s\right)\right\} \quad (61)$$

and it is one of the integers surrounding N.

Proof: Optimal strategy analysis of an N-policy two-phase $M^x/M/1$ queuing system with server startup and breakdowns[11].

7. Sensitivity Analyses

In this section, sensitivity analysis is performed on the optimum threshold N^* based on changes in the system parameters and the cost elements through numerical illustrations.

Let the batch size X have geometric distribution with mean batch size 1/p.

Then $a_j = P(X=j) = p(1-p)^{j-1}$, $0 < p < 1$, $j = 1,2,3,\ldots$ with the probability generating function $A(z) = pz/[1-(1-p)z]$ and $E(X) = A^1(1) = 1/p$, $E[X(X-1)] = A^{11}(1) = 2(1-p)/p^2$.

The expected number of customers in the system is given by

$$L(N) = \frac{\rho_2}{(1-\rho_2)} + \frac{k\rho_1}{(1-\rho_2)} + \frac{Y_N^{\,1}(1)P_{0,0,0}}{(1-\rho_2)} + \frac{k\lambda}{p(1-\rho_2)}\left(\frac{\lambda + \theta Y_N(1)}{\theta^2}\right)P_{0,0,0}$$

$$+\frac{\lambda^2 k\alpha_1}{p^2\beta\gamma\delta(1-\rho_2)}\left(1+\frac{\gamma}{\delta}+\frac{\delta}{\gamma}\right) + \frac{k\lambda^2\alpha_2}{p^2\mu\gamma\delta}\left(1+\frac{\gamma}{\delta}+\frac{\delta}{\gamma}\right)$$

$$+\frac{(2k\text{-}(k+1)p)\rho_2}{2p(1-\rho_2)}\left(\frac{\lambda+\theta Y_N(1)}{\theta}\right)P_{0,0,0} + \frac{(2k\text{-}(k+1)p)\rho_2(\rho_1+\rho_2)}{2p(1-\rho_2)}$$

$$+\frac{k\lambda\rho_2}{\rho(1-\rho_2)}\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)\left(\rho_2 - \frac{\lambda}{p\mu}\right) - \frac{k\lambda^2\alpha_2\rho_2}{p^2\mu\gamma\delta(1-\rho_2)}$$

where $\rho_1 = \frac{\lambda}{p\beta}\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)$, $\rho_2 = \frac{\lambda}{p\mu}\left(1+\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)$,

$$P_{0,0,0} = (1-\rho_1-\rho_2)\frac{\theta}{(\lambda+\theta Y_N(1))}.$$

The varying details of optimal threshold N* and the minimum cost $C_A(N^*)$ for specified values of the system parameters and the cost elements are presented in the following Tables.

From Table 1 it may be observed that (a) N* shows increasing trend for increase in the values of λ and μ, (b) $C_A(N^*)$ increases with increase in λ and decreases with increase in μ.

It can be seen from Table 2 that (a) N* is insensitive to the values of β and it decreases with increase in the values of θ, (b) with increase in the values of β, $C_A(N^*)$ decreases. Conversely it decreases with increase in the values of θ.

It can be observed from Table 3 that (a) N* shows decreasing trend with increase in values of C_h where as it is insensitive with increase in C_0 and (b) $C_A(N^*)$ increases with increase in the values of C_h and C_0.

Also, it is observed from the numerical computations (which are not presented) that (a) N* is insensitive to increase in the values of α_1, C_b and C_r and it increases with increase in the values of γ, C_m and C_s (b) $C_A(N^*)$ decreases

Table 1. The optimal $N^*$and minimum expected cost $C_A(N^*)$with various (λ, μ)

$\beta = 4, \theta = 3, \gamma = 3, m = 2, \alpha_1 = 0.5, \alpha_2 = 0.5, \delta = 2, C_h = 5, C_0 = 50, C_m = 200, C_b = 100, C_r = 40, C_s = 1000$

(λ,μ)	N^*	$C_A(N^*)$	(λ,μ)	N^*	$C_A(N^*)$
(0.2,3.5)	17	29.73	(0.4,1.5)	13	91.35
(0.3,3.5)	21	46.73	(0.4,2.0)	17	76.17
(0.4,3.5)	23	59.94	(0.4,2.5)	19	68.33
(0.5,3.5)	24	69.74	(0.4,3.0)	21	63.11
(0.6,3.5)	23	76.19	(0.4,3.5)	21	59.41

Table 2. The optimal N^* and minimum expected cost $C_A(N^*)$ with various (β,θ)

$\lambda = 0.4, \theta = 3, \mu = 3.5, \theta = 3, \gamma = 3, m = 2, \alpha_1 = 0.5, \alpha_2 = 0.5, \delta = 2, C_h = 5, C_0 = 50, C_m = 100, C_b = 100, C_r = 40, C_s = 1000$

(β,θ)	N^*	$C_A(N^*)$	(β,θ)	N^*	$C_A(N^*)$
(3.0,3.0)	15	13.33	(4.0,0.05)	29	65.29
(4.0,3.0)	15	11.85	(4.0,0.1)	23	42.15
(5.0,3.0)	15	10.95	(4.0,0.15)	21	33.20
(6.0,3.0)	15	10.36	(4.0,0.2)	19	28.33
(7.0,3.0)	15	9.93	(4.0,0.3)	17	23.15

Table 3. The optimal N^* and minimum expected cost $C_A(N^*)$with various (c_h,c_o)

$\lambda = 0.5, \mu = 3.5, \beta = 4, \theta = 3, \gamma = 3, m = 2, \alpha_1 = 0.5, \alpha_2 = 0.5, \delta = 2, C_0 = 50, C_m = 200, C_b = 100, C_r = 40, C_s = 1000$					
(c_h,c_o)	N^*	$C_A(N^*)$	(c_h,c_o)	N^*	$C_A(N^*)$
(3.0,50)	29	59.19	(5.0,30)	23	59.03
(4.0,50)	27	64.58	(5.0,40)	23	64.38
(5.0,50)	23	69.74	(5.0,50)	23	69.74
(6.0,50)	21	74.52	(5.0,60)	23	75.10
(7.0,50)	19	79.11	(5.0,70)	23	80.46

with increase in the values of γ, δ and C_r and increases with increase in the values of α_1, α_2 and C_m, C_b, C_s.

8. Conclusion

In this paper, some important performance measures are derived for the N-policy, $M^x/E_k/1$ queuing system with two phases of service, server startup, server breakdowns and delayed repair. As the convexity of the expected cost function cannot be proved theoretically, a heuristic approach is chosen to determine the optimal threshold N^*. Sensitivity analysis is performed between the optimal threshold N^*, and specific values of system parameters and cost elements for an arrival batch size distribution, geometric.

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