

Quotient Amenability of Banach Algebras

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Abstract

Let A be a Banach algebra and I be a non-zero closed two-sided ideal of A . We say that the Banach algebra A is I -quotient amenable if the quotient Banach algebra A/I is amenable. In this paper we study this notion and give a sufficient condition for I -quotient amenability. Also, we provide a characterization of I -quotient amenability whenever I has a bounded approximate identity. We prove that this notion may be coincide with amenability, then apply this result to give a new characterization for amenability of C^* -algebras. Finally, we give some results over the Fourier algebra.

Keywords: Amenability, C^* -Algebra, Fourier Algebra, Quotient Algebra

1. Introduction

In the literature of mathematics, an amenable group is a locally compact topological group G carrying a kind of averaging operation on bounded functions that is invariant under translation by group elements. The original definition, in terms of a finitely additive invariant measure (or mean) on subsets of G , was introduced by J. Von Neumann in 1929 in response to the Banach-Tarski paradox¹.

After that many authors worked on this notion. One of those notable works is, cohomology of Banach algebra² by B. E. Johnson. In this memoir Johnson state an interesting theorem that with this theorem the concept of amenability carried from groups to Banach algebras. Amenable Banach algebras have since proved themselves to be widely applicable in modern mathematical analysis; for example see^{3,4}.

In this paper we introduce a new concept of amenability that is a generalization of amenability in some sense. Indeed, we try to generating Banach algebras which they are amenable.

1.1 Preliminaries

In this section we give a brief outline of the definitions and known results. For further details one can refer to²⁻⁵.

Suppose that A is a Banach algebra and X is a Banach A -bimodule. A linear map $D: A \rightarrow X$ is a derivation if, for all $a, b \in A$,

$$D(ab) = D(a).b + a.D(b).$$

If $\in X$, the mapping D_x defined by $D_x(a) = a.x - x.a$ for all $a \in A$ is a derivation. We say that D_x is an inner derivation that implemented by x .

Let $Z^1(A, X)$ be the space of all continuous derivation from A to X and $N^1(A, X)$ be the space of all inner derivation from A to X . The quotient space $H^1(A, X) = Z^1(A, X)/N^1(A, X)$ is called the first cohomology group of A with coefficients in X .

If X is a Banach A -bimodule, X^* is a Banach A -bimodule with the following module actions:

$$\langle a.x^*, x \rangle = \langle x^*, x.a \rangle, \langle x^*.a, x \rangle = \langle x^*, a.x \rangle \quad (a \in A, x \in X, x^* \in X^*).$$

The projective tensor product of the Banach algebra A defined by

$$A \hat{\otimes} A = \left\{ \sum_{n=1}^{\infty} x_n \otimes y_n : \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty, x_n, y_n \in A \right\}$$

The projective tensor product $A \hat{\otimes} A$ is an A -bimodule with the module actions:

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$$a.(b \otimes c) = ab \otimes c, (b \otimes c).a = b \otimes ca \quad (a, b, c \in A).$$

Also, it is a Banach algebra with the product and norm defined by

$$(a \otimes c).(b \otimes d) = ab \otimes cd, \quad (a, b, c, d \in A),$$

$$\|d\| = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \|y_n\|, d = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}.$$

Let $\pi : A \hat{\otimes} A \rightarrow A$ be the bounded linear map defined by $\pi(a \hat{\otimes} b) = ab$. A bounded net (m_α) in $A \hat{\otimes} A$ is an approximate diagonal if, for all $a \in A$

$$\|a.m_\alpha - m_\alpha.a\|, \|a - a.\pi(m_\alpha)\| \rightarrow 0.$$

The Banach algebra A is amenable if, every bounded derivation $D : A \rightarrow X^*$ for each Banach A -bimodule X is inner.

The following theorem is one of the important characterization of amenability due to B. E. Johnson.

Theorem 1.1: Let A be a Banach algebra. Then A is amenable if and only if it has a bounded approximate diagonal.

Let A be a C^* -algebra. We say that A is nuclear if, for all C^* -algebra B , there exists only one C^* -norm on $A \otimes B$.

Theorem 1.2 (Connes-Haagerup): Let A be a C^* -algebra. Then A is amenable if and only if A is nuclear.

Proof: See [4, Corollary 6.5.12].

Let G be a locally compact group. We say that G is an amenable group.

Exists a positive functional $m \in L^1(G)^{**}$ with $\|m\| = 1$ and $m(L_x f)$ for all $x \in G$ and $f \in L^1(G)^*$ where $L_x f(y) = f(x^{-1}y)$ for each $y \in G$. The class of amenable groups includes all compact groups and abelian groups, however the discrete free group F_2 on two generators is not amenable. The following theorem is due to B. E. Johnson and its proof is in². This theorem is fundamental in the theory of amenable Banach algebras.

Theorem 1.3 (Johnson): Let G be a locally compact group. Then $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable group.

We know that if a Banach algebra A is amenable, in some sense it is a small Banach algebra. Because amenable Banach algebras have bounded approximate diagonal by

Theorem 1.1. Hence, in this paper with use of an ideal of A , we try to construct a new Banach algebra which is smaller than A . Then we study the amenability properties of this new Banach algebra.

In the following section, we give our main definition. Then give some examples that shows this notion provide a class of Banach algebras which is larger than the class of the amenable Banach algebras. In the sequel of the section under some conditions we characterize this notion and then provide a sufficient condition in general. We show that for C^* -algebras, the notion of I-quotient amenability and amenability coincide when I is a nuclear closed ideal of A . Then we apply this result to give a new characterization for amenability of C^* -algebras.

In Section 3, we give some results over the Fourier algebra and show that for each locally compact group G , there exists a closed ideal I in $A(G)$; the Fourier algebra, such that $A(G)$ is I-quotient amenable.

2. Quotient Amenability

We commence this section with an important theorem of the amenability of Banach algebras that motivated us for the main definition of this paper.

Theorem 2.1: Let A be an amenable Banach algebra and I be a closed two-sided ideal of A .

- (1) The quotient Banach algebra $\frac{A}{I}$ is amenable.
- (2) If I has a bounded approximate identity, then it is amenable.

Proof: See [4, Corollary 2.3.2, Theorem 2.3.7].

By the first part of the above theorem if A is not amenable, then we do not know in general, is $\frac{A}{I}$ amenable or not? So, we focus on Banach algebras for which some of its quotients are amenable and give our main definition as follows. This definition will be used to generate amenable Banach algebras from non-amenable Banach algebras.

Definition 2.2: Let A be a Banach algebra and I be a non-zero closed two-sided ideal of A . We say that A is I-quotient amenable if, the quotient Banach algebra A/I is amenable and say A is quotient amenable if, for all closed two-sided non-trivial ideal I of A , the Banach algebra A is I-quotient amenable.

Every amenable Banach algebra is an I-quotient amenable Banach algebra for all closed two-sided ideal I, but there exist I-quotient amenable Banach algebras which are not amenable.

Recall that for a Banach algebra A, $\Delta(A)$ denotes the space of all characters of A, i.e., all non-zero linear and multiplicative map from A into $\mathbb{C}$ (See⁶ for further details).

Example 2.3: Suppose that A and B are Banach algebras which A is not amenable and B is amenable. Also, let $\theta \in \Delta(B)$. Then the θ -Lau product, denoted by¹² that defined as the set $A \times B$ equipped with the multiplication.

$$(a, b)(\acute{a}, \acute{b}) = (a\acute{a} + \theta(b)\acute{a} + \theta(\acute{b})a, b\acute{b}) \quad (a, \acute{a} \in A, \acute{b} \in B)$$

and the norm, $\|(a, b)\| = \|a\| + \|b\|$ is a Banach algebra.

We know that A is a closed two-sided ideal of C and $C/A \cong B$ as Banach algebras.

So, C/A is an amenable Banach algebra, but C is not amenable, because¹² is a strongly splitting extension of B, it is amenable if and only if both A and B are amenable⁷.

Example 2.4: Let G be a locally compact group that is not amenable and I be a closed two-sided ideal of $L^1(G)$ with finite codimension. By [3, Corollary 3.3.27], $L^1(G)/I$ is $*$ -isomorphic to a finite dimensional C^* -algebra. Therefore, by⁸ and Connes-Haagerup's Theorem, $L^1(G)$ is I-quotient amenable. But by Johnson's Theorem $L^1(G)$ is not amenable.

Remark 2.5: Note that all finite dimensional Banach algebras are not amenable. Because, finite dimensional amenable Banach algebras are unital. Also, each unital Banach algebra necessarily is not amenable. To see an example consider $L^1(\mathbb{F}_2)$ for which $\mathbb{F}_2$ is the discrete free group on two generators that is not amenable. So, $L^1(\mathbb{F}_2)$ is unital but it is not amenable.

Suppose that A is a Banach algebra, I is a closed two-sided ideal of A and X is a Banach A/I-bimodule. Then X is a Banach A-bimodule with the following module actions:

$$a \circ x = (a + I) \cdot x, \quad x \circ a = x(a + I) \quad (a, b \in A, x \in X).$$

But with use of θ -Lau product Banach algebras as in Example 2.3, we can give a Banach A-bimodule X which is not a Banach A/I-bimodule. Indeed, let A and B be two Banach algebras, $\theta \in \Delta(B)$ and X be a Banach A-bimodule which is not a Banach B-bimodule. Then X is a Banach see¹² B-bimodule with the following module actions:

$$(a, b).x = a.x, \quad x.(a, b) = x.a \quad ((a, b) \in C, x \in X).$$

But if we take $I = A$, then X is not a Banach $C/I \cong B$ -bimodule.

The following theorem characterize I-quotient amenability when I has a bounded approximate identity.

Theorem 2.6: Let A be a Banach algebra and I be a closed two-sided ideal with a bounded approximate identity. Then the following are equivalent;

- (1) The Banach algebra A is I-quotient amenable.
- (2) For each Banach A/I-bimodule X, every continuous derivation $D : A \rightarrow X^*$ is inner.

Proof: $1 \Rightarrow 2$): Let X be a Banach A/I-bimodule and let $D : A \rightarrow X^*$ be a continuous derivation.

Define a map $\tilde{D} : A/I \rightarrow X^*$ as follows:

$$\tilde{D}(a + I) = D(a) \quad (a \in A).$$

Now, we claim that $\tilde{D}$ is well-defined. Let $a, b \in A$ and $a + I = b + I$, so $a - b \in I$.

By Cohen factorization Theorem we have $I = I^2$. Therefore, there exists $c, d \in I$ such that

Since D is a derivation we have

$$D(a - b) = D(cd) = D(c).d + c.D(d).$$

But, for every $x \in X$ we have

$$\langle D(c).d, x \rangle = \langle D(c), d \circ x \rangle = \langle D(c), (d+1).x \rangle = 0.$$

Similarly, $\langle c.D(d), x \rangle = 0$. Hence, $D(a-b)=0$. So, $\tilde{D}(a + I) = \tilde{D}(b + I)$.

Now, if we show that $\tilde{D}$ is a derivation, I-quotient amenability of A yields D is inner and this completes the proof.

Let $a, b \in A$ and $x \in X$. Then we have

$$\begin{aligned} & \langle \tilde{D}(a + I).(b + I), x \rangle + \langle (a + I).\tilde{D}(b + I), x \rangle \\ &= \langle D(a).(b + I), x \rangle + \langle (a + I).D(b), x \rangle \\ &= \langle D(a), b \circ x \rangle + \langle D(b), x \circ a \rangle \\ &= \langle D(a).b, x \rangle + \langle a.D(b), x \rangle \\ &= \langle D(a.b), x \rangle. \end{aligned}$$

Therefore, $\tilde{D}$ is a derivation from A/I into X^* .

$2 \Rightarrow 1$): Let $D : A/I \rightarrow X^*$ be a continuous derivation where X is a Banach A/I-bimodule. Define $\tilde{D} : A/I \rightarrow X^*$ by $\tilde{D}(a) = D(a + I)$ for all $a \in A$.

In view of the definition of module actions of X , the map $\tilde{D}$ is a continuous derivation. Now, from the hypothesis we conclude that D is inner. Hence, A is I -quotient amenable. $\square$

In the sequel, we give a sufficient condition for I -quotient amenability without any conditions on I .

Let $(A^{**}, \square)$ be the second dual of the Banach algebra A that equipped with the first Arens product⁹. F. Gourdeau¹⁰ proved that A is amenable if A^{**} is amenable.

Recall that for a closed subspace I of A

$$I^\perp = \{f \in A^* : f(a) = 0 \quad (a \in I)\}.$$

Theorem 2.7: Let $((I^\perp)^*, \square)$ be an amenable Banach algebra for closed nontrivial two-sided ideal I of A . Then A is I -quotient amenable.

Proof: We know that if I is a closed subspace of A , then

$$\left(\frac{A}{I}\right)^* \cong I^\perp \text{ as Banach spaces via the mapping } H: a^* \rightarrow a^* \pi$$

where $\pi: A \rightarrow \frac{A}{I}$ is the natural homomorphism.

We show that the following mapping is an isometric isomorphism of Banach algebras:

$$((I^\perp)^*, \square) \rightarrow \left(\left(\frac{A}{I} \right)^{**}, \square \right)$$

$$a^{**} \rightarrow a^{**} H \quad (a^{**} \in (I^\perp)^*).$$

Let $a_1^{**}, a_2^{**} \in (I^\perp)^*, a^* \in \left(\frac{A}{I}\right)^*$ and $a, b \in A$. Then

$$\begin{aligned} ((a_1^{**} \square a_2^{**}) H)(a^*) &= a_1^{**} (a_2^{**} a^* \pi), \\ (a_2^{**} a^* \pi)(a) &= a_2^{**} ((a^* \pi) a), \\ ((a^* \pi) a)(b) &= a^* (\pi(a) b). \end{aligned}$$

On the other hand,

$$\begin{aligned} ((a_1^{**} H) a_2^{**} H)(a^*) &= a_1^{**} H((a_2^{**} H) a^*) = a_1^{**} ((a_2^{**} H) a^*) \pi, \\ ((a_2^{**} H) a^*) \pi(a) &= a_2^{**} (a^* \pi(a)) \pi, \\ ((a^* \pi(a)) \pi)(b) &= a^* (\pi(a) \pi(b)). \end{aligned}$$

Since for the natural homomorphism $\pi, \pi(ab) = \pi(a)\pi(b)$, from the above relations we conclude that the mapping (2.1) is multiplicative. The other properties follows from this fact that H is an isometric isomorphism.

Hence, $\left(\frac{A}{I}\right)^* \cong I^\perp$ as Banach algebra. Therefore,

$\left(\frac{A}{I}\right)^{**}$ is an amenable Banach algebra. So, Gourdeau's

Theorem implies that $\frac{A}{I}$ is amenable. Hence, A is I -quotient amenable.

The following theorem shows that the amenability and I -quotient amenability coincide for some of Banach algebras.

Theorem 2.8: Let A be a C^* -algebra and I be a closed two-sided nuclear ideal of A . Then A is I -quotient amenable if and only if A is amenable.

Proof. Let A be I -quotient amenable. So, A/I is nuclear by Connes-Haagerup Theorem, since A/I is a C^* -algebra by [8 Theorem 3.1.4]. In view of [11 Corollary 3.4] we know that A is nuclear if and only if I and A/I are both nuclear.

Therefore, A is nuclear and hence A is amenable.

We must note that in a C^* -algebra A , there exist a lot of ideals which are nuclear. As an example [8, Theorem 6.3.9] each finite-dimensional ideal in A is nuclear or [8, Theorem 6.4.15] every commutative closed two-sided ideal in A is nuclear.

On the other hand, we know that each closed ideal in a C^* -algebra has a bounded approximate identity. Hence, Theorems 2.8 and 2.1 implies the following corollary.

Corollary 2.9: Let A be a C^* -algebra. Then A is amenable if and only if for some non-zero finite-dimensional ideal I of A , $H^1(A, X) = \{0\}$ for each Banach A/I -bimodule X .

Remark 2.10: In the above theorem, the class of Banach modules X for which becomes smaller. Since each Banach A/I -bimodule X is a Banach A -bimodule, but the converse is not true in general.

Now, we compare the quotient amenability of the Banach algebra A by two ideals $I \subseteq J$.

Theorem 2.11: Let I, J be two closed two-sided ideals of A where $I \subseteq J$. Then

- (1) A is J -quotient amenable if A is I -quotient amenable.
- (2) A is I -quotient amenable if A is J -quotient amenable and J is I -quotient amenable.
- (3) J is I -quotient amenable if it has a bounded approximate identity and A is I -quotient amenable.

Proof (1): Let A be I -quotient amenable. So, there exists a bounded approximate diagonal (m'_a) such that

$$\|(a+1).m_a - m_a.(a+1)\|, \|(a+1) - (a+1)\pi(m_a)\| \rightarrow 0 \quad (a \in A).$$

Let $m_a = \sum_{i=1}^{\infty} (a_i^a + 1) \otimes (b_i^a + 1)$ and $m'_a = \sum_{i=1}^{\infty} (a_i^a + J) \otimes (b_i^a + J)$. we show that the net (m'_a) is a bounded approximate diagonal for A/J . First we have

$$\|(a+J) - (a+J)\pi(m'_a)\| \leq \|(a+I) - (a+I)\pi(m_a)\| \rightarrow 0 \quad (a \in A).$$

The boundedness of the net (m'_a) is clear from the inequality,

$$\|(a+J) \otimes (b+J)\| \leq \|(a+I) \otimes (b+I)\|, \quad (a, b \in A).$$

On the other hand, if $d_a = (a+I).m_a - m_a.(a+I) = \sum_{j=1}^{\infty} (x_j^a + I) \otimes (y_j^a + I)$, is a representation of d_a , $d'_a = (a+J).m'_a - m'_a.(a+J) = \sum_{j=1}^{\infty} (x_j^a + J) \otimes (y_j^a + J)$ is a representation of d'_a .

Now, let $\epsilon > 0$. In view of the above relations and definition of the projective tensor product norm, there exists a representation $\sum_{j=1}^{\infty} (x_j^a + I) \otimes (y_j^a + I)$ of $d_a = (a+I).m_a - m_a.(a+I)$ such that

$$\sum_{j=1}^{\infty} \|x_j^a + I\| \|y_j^a + I\| < \|d_a\| + \epsilon$$

Hence,

$$\|d'_a\| \leq \sum_{j=1}^{\infty} \|x_j^a + J\| \|y_j^a + J\| \leq \sum_{j=1}^{\infty} \|x_j^a + I\| \|y_j^a + I\| < \|d_a\| + \epsilon.$$

So, $\lim_a \|d'_a\| \leq \epsilon$. Therefore, $\lim_a \|(a+J).m'_a - m'_a.(a+J)\| = 0$.

Hence, A/J has a bounded approximate diagonal, therefore A/J is amenable.

(2): Suppose that A is J -quotient amenable and J is I -quotient amenable (note that I is a closed two-sided ideal of J). We know that the mapping

$$F: \frac{A/J}{J/J} \rightarrow A/J$$

$$(a+I) + J/I \rightarrow a+J \quad (a \in A)$$

is an isomorphism that is norm increasing, i.e., for all $a \in A$ we have

$$\|(a+I) + J/I\| \leq \|F((a+I) + J/I)\|.$$

Since A is J -quotient amenable, it has a bounded approximate diagonal (m'_a) where $m_a = \sum_{i=1}^{\infty} (a_i^a + J) \otimes (b_i^a + J)$.

With a similar proof as above, if $d_a = (a+J).m_a - m_a.(a+J) = \sum_{j=1}^{\infty} (x_j^a + J) \otimes (y_j^a + J)$ is a representation of d_a ,

$$\begin{aligned} d'_a &= ((a+I) + J/I).m'_a - m'_a.((a+I) + J/I) \\ &= \sum_{j=1}^{\infty} ((x_j^a + I) + J/I) \otimes ((y_j^a + I) + J/I), \end{aligned}$$

is a representation d'_a of where Obviously, $m'_a \sum_{i=1}^{\infty} ((a_i^a + I) + J/I) \otimes ((b_i^a + I) + J/I)$. Obviously,

$$\begin{aligned} &\|((a+I) + J/I) - ((a+I) + J/I)\pi(m'_a)\| \\ &\leq \|(a+J) - (a+J)\pi(m_a)\| \rightarrow 0 \quad (a \in A). \end{aligned}$$

The boundedness of (m'_a) is clear from the following inequality,

$$\begin{aligned} &\|((a+I) + J/I) \otimes ((b+J) + J/I)\| \\ &\leq \|(a+J) \otimes (b+J)\| \quad (a, b \in A). \end{aligned}$$

Let $\epsilon > 0$. There exists a representation $\sum_{j=1}^{\infty} (x_j^a + J) \otimes (y_j^a + J)$ of d such that

$$\sum_{j=1}^{\infty} \|x_j^a + I\| \|y_j^a + I\| < \|d_a\| + \epsilon. \text{ Now, we have}$$

$$\begin{aligned} \|d'_a\| &\leq \sum_{j=1}^{\infty} \left\| (x_j^a + I) + \frac{J}{I} \left\| (y_j^a + I) + \frac{J}{I} \right\| \right\| \\ &\leq \sum_{j=1}^{\infty} \left\| x_j^a + I \right\| \left\| y_j^a + I \right\| < \|d_a\| + \epsilon. \end{aligned}$$

So, (m'_α) is a bounded approximate diagonal for $\frac{A}{J/I}$. Hence, it is an amenable Banach algebra.

On the other hand, $\frac{J}{I}$ is an amenable Banach algebra and it is a closed two sided ideal of A/I . Hence A/I is amenable.

(3): In this case J/I is an ideal of A/I that has a bounded approximate identity, so is amenable.

3. Application over the Fourier Algebra

Let G be a locally compact group and $\lambda: G \rightarrow B(L^2(G))$ be the left regular representation of G , i.e.,

$$(\lambda(x)f)(y) = f(x^{-1}y) \quad (x, y \in G, f \in L^2(G)).$$

Let $A(G)$ be the set of all functions $u \in C_0(G)$; the space of all continuous function with zero limit at infinity, of the form $u(x) = \langle \lambda(x)f, g \rangle$ where $f, g \in L^2(G)$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(G)$. An element of $A(G)$ can be represented as $u = f * \check{g}$ where $f, g \in L^2(G)$ and $\check{g}(x) = g(x^{-1})$. Now, for $u \in A(G)$.

$$\|u\|_{A(G)} = \inf \left\{ \|f\|_2 \|g\|_2 : f, g \in L^2(G), u = f * \check{g} \right\}$$

It is well known that $\|\cdot\|_{A(G)}$ is a norm on $A(G)$ which makes this space into a commutative Banach algebra with pointwise multiplication called the Fourier algebra of G .

For each closed subset H of G , define

$$I(H) = \{u \in A(G); u(x) = 0 \quad \forall x \in H\}.$$

Obviously, $I(H)$ is a closed ideal in $A(G)$ (See¹² for some interesting results over $I(H)$).

Now, we give the following theorem which is a main result of this section.

Theorem 3.1: Let G be a locally compact group and H be a closed subgroup of G such that has an abelian subgroup of finite index. Then $A(G)$ is $I(H)$ -quotient amenable. Especially for each closed abelian subgroup H of G , $A(G)$ is $I(H)$ quotient amenable.

Proof: Let H be a closed subgroup such that has an abelian subgroup of finite index. By [13, Lemma 3.8]

we know that $A(H)$ is isometrically isomorphic to $A(G)/I(H)$.

On the other hand, by [14, Theorem 7.11] we know that $A(H)$ is amenable if and only if H has an abelian subgroup of finite index. Therefore, $A(H)$ and so $A(G)/I(H)$ is amenable and this completes the proof.

Each locally compact group G has an abelian closed subgroup $H = \{e\}$ where e is the identity of G . So, we have the following corollary.

Corollary 3.2: Let G be a locally compact group. There exists a non-zero ideal I of $A(G)$ such that $A(G)$ is I -quotient amenable.

The following example give a group G and a closed abelian subgroup H of G such that $A(G)$ is not amenable, but $A(G)$ is $I(H)$ -quotient amenable.

Example 3.3: Let G be the $\mathbf{ax} + \mathbf{b}$ group whose underlying manifold is $(0, \infty) \times \mathbb{R}$ and its group action is

$$(a, b)(c, d) = (ac, b + ad).$$

Clearly, $H = \{(a, 0) : a > 0\}$ is a closed abelian subgroup of G . Therefore, $A(G)$ is $I(H)$ -quotient amenable by Theorem 3.1. But G does not have any abelian subgroup of finite index. Hence, $A(G)$ is not amenable.

To see an example of a non- I -quotient amenable Banach algebra, let K be a locally compact group for which contain the $\mathbf{ax} + \mathbf{b}$ group G as a closed subgroup.

Then, $A(K)$ is not $I(G)$ -quotient amenable.

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