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Derivation of Block Methods for Solving Second Order Ordinary Differential Equations Directly using Direct Integration and Collocation Approaches

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Abstract

This paper presents two block methods for solving second order ordinary differential equations directly. These two methods are capable of computing the numerical solutions at several points simultaneously. In deriving these methods, two techniques are employed namely direct integration and collocation approach. The advantages and drawbacks of each method are also discussed.

Keywords: Block Method, Collocation, Direct Integration, Direct Method, Initial Value Problems, Ordinary Differential Equations

1. Introduction

In the subsequent discussion we consider solving the following second order Ordinary Differential Equation (ODE)

$$y'' = f(x, y, y'),\ y(a) = y_0,\ y'(a) = y'_0,\ a \le x \le b \qquad (1)$$

There are two techniques available for solving Equation (1). The first technique is to reduce (1) to a system of first order equations and then solve it using first order ordinary differentials (ODEs) methods. This reduction method is known to have some setbacks which include: wastage of computer time, a lot of human effort and computer program developed to examine the accuracy of the method is usually found to be complicated[1]. These methods are very well.

Another approach is to solve (1) directly as suggested by some researhers[2–3,5,8–9]. This method solves higher order initial value problems of ODEs without going through the process of reduction. A lot of scholars such as [1,6,7] also proposed numerical methods for solving (1) directly whereby predictor-corrector mode was made use in implementing the methods. It is observed that this predictor-corrector method involves many functions to evaluate which amount to computational burdens that always affect the accuracy of the methods in terms of error and also increase computational time. In addition, this method computes the numerical solution at one point at a time.

The aim of this paper is to derive block methods without predictors in such a way that the numerical solutions can be computed at several points at a time. In deriving the methods, two techniques will be employed namely direct integration and collocation method. The

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details derivations of the methods are described in the following sections.

2. Derivation of R-Point Implicit Block Method using Direct Integration

Let $x_{n+j} = x_n + jh,\ j = 1,2,...,R$. Integrating (1) once and taking the limit from x_n to x_{n+j} gives

$$y'_{n+j} - y'_n = \int_{x_n}^{x_{n+j}} f(x,y,y')\,dx.$$

Replacing $f(x,y,y')$ with a polynomial

$$P_{k+1,n+j}(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m f_{n+j}$$

which interpolates $f(x,y,y')$ at the $k+1$ back values namely x_{n+j-i} for $i = 0,1,2,...,k$ gives the following result

$$y'_{n+j} - y'_n = \int_{x_n}^{x_{n+j}} \sum_{m=0}^{k} (-1)^m \binom{-s_j}{m} \nabla^m f_{n+j}\,dx. \qquad (2)$$

where

$$s_j = \frac{x - x_{n+j}}{h}.$$

Changing the limit integration and substituting $dx = hds_j$ in (2) yields which leads to

$$y'_{n+j} = y'_n + h\sum_{m=0}^{k} \alpha^{(1)}_{j,m} \nabla^m f_{n+j} \qquad (3)$$

where

$$\alpha^{(1)}_{j,m} = (-1)^m \int_{-j}^{0} \binom{-s_j}{m} ds_j.$$

Define the generating function $G^{(1)}_j(t)$as follows

$$\begin{aligned} G^{(1)}_j(t) &= \sum_{m=0}^{k} \alpha^{(1)}_{j,m} t^m \\ &= \sum_{m=0}^{\infty} (-t)^m \int_{-j}^{0} \binom{-s_j}{m} ds_j \\ &= \int_{-j}^{0} \sum_{m=0}^{\infty} (-t)^m \binom{-s_j}{m} ds_j \\ &= \int_{-j}^{0} e^{-s_j \log(1-t)} ds_j \end{aligned} \qquad (4)$$

To approximate the value of y at x_{n+j}, we integrate (1) twice and taking the limit from x_n to x_{n+j} as below

$$y_{n+j} - y_n - (jh)y'_n = \int_{x_n}^{x_{n+j}} (x_{n+j} - x) f(x,y,y')dx. \qquad (5)$$

Substituting $f(x,y,y')$ with a polynomial

$$P_{k+1,n+j}(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m f_{n+j} \quad \text{in (5) gives}$$

$$y_{n+j} - y_n - (jh)y'_n = \int_{x_n}^{x_{n+j}} (x_{n+d} - x) \sum_{m=0}^{k} (-1)^m \binom{-s_j}{m} \nabla^m f_{n+j} \qquad (6)$$

Next, replace $dx = hds_j$ and change the limit of integration in (6) to obtain

$$y_{n+j} - y_n - (jh)y'_n = \int_{x_n}^{x_{n+j}} \sum_{m=0}^{k} (-1)^m \binom{-s_j}{m} \nabla^m f_{n+j}(\text{-}hs_j)hds_j \qquad (7)$$

which can be written as

$$y_{n+j} = y_n + (jh)y'_n = \sum_{m=0}^{k} \alpha^{(2)}_{j,m} \nabla^m f_{n+j} \qquad (8)$$

where

$$\alpha^{(2)}_{j,m} = (-1)^m \int_{-j}^{0} \sum_{m=0}^{\infty} (-s_j) \binom{-s_j}{m} ds_j \qquad (9)$$

Let the generating functions $G^{(2)}_j(t)$defined as follows

$$\begin{aligned} G^{(2)}_j(t) &= \sum_{m=0}^{k} \alpha^{(2)}_{j,m} t^m \\ &= \sum_{m=0}^{\infty} (-t)^m \int_{-j}^{0} (-s_j) \binom{-s_j}{m} ds_j \\ &= \int_{-j}^{0} (-s_j) \sum_{m=0}^{\infty} (-t)^m \binom{-s_j}{m} ds_j \\ &= \int_{-j}^{0} (-s_j) e^{-s_j \log(1-t)} ds_j \end{aligned} \qquad (10)$$

The constant step size formulations of implicit k-step method corresponding to (3) and (8) are

$$y'_{n+j} = y'_n + h\sum_{m=0}^{k} \beta^{(j,1)}_{k,m} f_{n+j-m} \qquad (11)$$

and

$$y_{n+j} = y_n + (jh)y'_n = h^2 \sum_{m=0}^{k} \beta^{(j,2)}_{k,m} \nabla^m f_{n+j-m} \qquad (12)$$

respectively. The coefficients $\{\beta_{k,m}^{(j,p)} | \ m = 0,1,\ldots,k \text{ and } p = 1,2 \}$ are well known and defined by Shampine and Gordon (1975) as follows

$$\beta_{k,m}^{(j,p)} = (-1)^m \sum_{r=m}^{k} \binom{r}{m} \alpha_{j,m}^{(p)} \tag{13}$$

Hence, our main task now is to determine the integration coefficients $\alpha_{j,m}^{(p)}$. Once $\alpha_{j,m}^{(2)}$ are known, (11) and (12) can be solved by applying (13).

From (10), we have

$$G_j^{(2)}(t) = \int_{-j}^{0} (-s_j) e^{-s_j \log(1-t)} ds_j \tag{14}$$

Applying integration by parts on the right term of (14) gives the following relationship

$$G_j^{(2)}(t) = \frac{j(1-t)^j - G_j^{(1)}(t)}{\log(1-t)} \tag{15}$$

which is equivalent to

$$\sum_{m=0}^{\infty} \alpha_{j,m}^{(2)} t^m = \frac{j(1-t)^j - \sum_{m=0}^{\infty} \alpha_{j,m}^{(2)} t^m}{\log(1-t)} \tag{16}$$

Substituting

$$\log(1-t) = \left[-\left(t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \ldots. + \frac{1}{m}t^m + \ldots \right) \right]$$

in (16) and then expanding, rearranging and equating coefficient of t^m gives the following solution

$$\alpha_{j,0}^{(2)} = \alpha_{j,1}^{(1)} + j^p$$

$$\alpha_{j,m}^{(2)} = \alpha_{j,m+1}^{(1)} - j \binom{j}{m+1} (-1)^{m+1} - \sum_{r=0}^{m-1} \frac{\alpha_{j,r}^{(2)}}{(m-1)+2-r} \ , \ m = 1,2,\ldots, j-1$$

$$\alpha_{j,m}^{(2)} = \alpha_{j,m+1}^{(1)} - \sum_{r=0}^{m-1} \frac{\alpha_{j,r}^{(2)}}{(m-1)+2-r} \ , \ m = j, j+1, \ldots, k+n-p$$

Integrating (1) once i.e. p=1 produces the following results

$$\alpha_{j,0}^{(1)} = j$$

$$\alpha_{j,m}^{(1)} = -\binom{j}{m+1} (-1)^{m+1} - \sum_{r=0}^{m-1} \frac{\alpha_{j,r}^{(1)}}{(m-1)+2-r} \ , \ m = 1,2,\ldots, j-1$$

$$\alpha_{j,m}^{(1)} = -\sum_{r=0}^{m-1} \frac{\alpha_{j,r}^{(1)}}{(m-1)+2-r} \ , \ m = j, j+1, \ldots, k+n$$

3. Derivation of the K-step Block Method using Collocation

In developing this method, we assume a power series of the form

$$y(x) = \sum_{j=0}^{k+2} a_j x^j \tag{17}$$

as an approximate solution to equation (1). Equation (18) is differentiated twice and it gives

$$y'(x) = \sum_{i=1}^{k+2} j a_j x^{j-1} \tag{18}$$

$$y''(x) = \sum_{j=2}^{k+2} j(j-1) a_j x^{j-2} = f(x, y, y') \tag{19}$$

Equations (17) and (19) are interpolated and collocated at the points $x = x_{n+i}$,$i = (k-3)(1)(k-2)$ and $x = x_{n+i}$,$i = 0(1)k$ respectively.

The interpolation and collocation equations at the selected grid points give a system of nonlinear equation of the for

$$\begin{pmatrix} 1 & x_{n+k-3} & x_{n+k-3}^2 & x_{n+k-3}^3 & \cdots & x_{n+k-3}^{k+2} \\ 1 & x_{n+k-2} & x_{n+k-2}^2 & x_{n+k-2}^3 & \cdots & x_{n+k-2}^{k+2} \\ 0 & 0 & 2 & 6x_n & \cdots & (k+2)(k+1)x_n^k \\ 0 & 0 & 2 & 6x_{n+1} & \cdots & (k+2)(k+1)x_{n+1}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & 6x_{n+k} & \cdots & (k+2)(k+1)x_{n+k}^k \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{k+2} \end{pmatrix} = \begin{pmatrix} y_{n+k-} \\ y_{n+k-} \\ f_n \\ f_{n+} \\ \vdots \\ f_{n+k} \end{pmatrix} \tag{20}$$

Gaussian elimination method is applied to find the values of the unknown variables (a's) in (20) which then are substituted into equation (17) to produce a continuous implicit scheme of the form

$$y(z) = \sum_{j=k-3}^{k-2} \alpha_j(z) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(z) f_{n+j} \tag{21}$$

where

$$z = \frac{x - x_{n+k-1}}{h},$$

$\alpha_j(z)$ and $\beta_j(z)$ are coefficients to be determined.

Equation (21) is evaluated at the non-interpolating points $x = x_n, x_{n+1}, \ldots.. x_{n+k-4}, x_{n+k-1}, x_{n+k}$ to produce the discrete schemes.

The derivative of (21) is evaluated at all the grid points $x = x_{n+v}, v = 0(1)k$ to give the derivative of the discrete schemes. The schemes and its derivatives are combined in a matrix form to give a block of the form

$$A^{(0)}Y_{N+1} = A^{(1)}Y_N + hB^{(1)}Y_N^{'} + +h^2(C^{(0)}F_{N+1} + D^{(1)}F_N) \qquad (22)$$

where $A^{(0)}, A^{(1)}, B^{(1)}, C^{(0)}, D^{(1)}$ are squared matrices,

$$Y_{N+1} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \\ y_{n+k} \end{pmatrix}, Y_N = \begin{pmatrix} y_{n-(k-1)} \\ y_{n-(k-2)} \\ y_{n-(k-3)} \\ \vdots \\ y_n \end{pmatrix}, Y_n' = \begin{pmatrix} y'_{n-(k-1)} \\ y'_{n-(k-2)} \\ y'_{n-(k-3)} \\ \vdots \\ y'_n \end{pmatrix},$$

and

$$F_{N+1} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ \vdots \\ f_{n+k} \end{pmatrix}, F_N = \begin{pmatrix} f_{n-(k-1)} \\ f_{n-(k-2)} \\ f_{n-(k-3)} \\ \vdots \\ f_n \end{pmatrix}$$

4. Discussion and Conclusion

Two different approaches namely direct integration and collocation method for developing numerical methods for the solution of general second order initial value problems have been considered in this paper. Although the derivation using direct integration method is more complicated if compared to its counterpart, this method is able to generalize the formulation of the integration coefficients for any k back values used at any point. On the other hand, the derivation of block method using collocation method seems simpler. But this approach fails to generalize the formulation of unknowns a's to any step length k. It is due to the fact that the order of differential equation that determines the number of interpolation points. Also, the order of the approximated power series is determined by the number of interpolation point.

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