Characterization of Bipolar Fuzzy Soft Γ-Semigroups

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Abstract

This paper deals with the idea of bipolar fuzzy soft sets applied to the ideal theory of Γ-semigroups. We have introduced the concept of bipolar fuzzy soft Γ-subsemigroup and bipolar fuzzy soft Γ-ideals in a Γ-semigroup. It is proved that the extended union, extended intersection, restricted union and restricted intersection of two same kind bipolar fuzzy soft Γ-ideals over a Γ-semigroup produced a same kind's bipolar fuzzy soft Γ-ideal. Also the "AND" and "OR" operations of two bipolar fuzzy soft Γ-ideals produced a same type's bipolar fuzzy soft Γ-ideal. It is also proved that the collection of all bipolar fuzzy soft Γ-ideals over a Γ-semigroup forms a complete distributive lattice with these special unions and intersections.

Keywords: Bipolar Fuzzy Soft Set, Bipolar Fuzzy Soft Γ-subsemigroup, Bipolar Fuzzy Soft Γ-ideals.
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1. Introduction

After the introduction of the classical notion of fuzzy sets by Zadeh[26] in 1965, many scientists used fuzzy sets in different fields of science. The use of fuzzy sets in algebraic structures was done by Rosenfeld[18] in 1971. He defined fuzzy subgroups and discussed their important properties. The idea of fuzzy ideals, fuzzy bi-ideals in semigroup was given by Kuroki[9–11]. With the passage of time researchers introduced many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar fuzzy sets etc. Zhang[1, 4] introduced the notion of bipolar fuzzy sets and used it for modeling and decision analysis. Lee[12, 13] used the term bipolar valued fuzzy sets and applied it to the algebraic structures. Soft set theory was initiated by Molodtsov[16] in 1999. In[14], Maji and his co-authors defined some basic operations on soft sets. Ali and his co-authors modified the binary operations defined by Maji et al.

Maji et al.[15] combined the concept of fuzzy set and soft set and proposed the notion of fuzzy soft set in 2001 and defined the basic operations of these fuzzy soft sets. The results[15] were improved by Ahmad and Kharal. Aktas and Cagman[3], introduced the concept of fuzzy soft groups

which was extended by Aygunoglu and Aygun[6]. The concept of fuzzy soft semigroups and fuzzy soft ideals were given by Yang[25]. The concepts of fuzzy bipolar soft sets and bipolar fuzzy soft sets has been introduced by Naz and Shabir[17] recently. They defined their special union and special intersection and also showed that the both notions are equivalent. Aslam et al.[5], also worked on bipolar fuzzy soft sets and their special union and special intersection.

The notion of Γ-semigroup was introduced by Sen and Saha[23] in 1986. They studied regular Γ-semigroup and Γ-group and established some relations between them (see also[19, 20]). The notion of bi-ideals in Γ-semigroup was introduced by Chinram and Jirojkul[7].

The notion of fuzzy ideals in Γ-semigroup was introduced by Sardar et al.[21, 22], they also worked on fuzzy bi-ideals and fuzzy quasi-ideals of Γ-semigroups. William et al.[24] and Faisal et al.[8], also worked on the fuzzy ideal theory of Γ-semigroups. Akram et al.[2] introduced the notion of fuzzy soft Γ-semigroups and characterized them by their fuzzy soft Γ-ideals.

In this paper, we have defined different bipolar fuzzy soft Γ-ideals over a Γ-semigroup. Also it has been proved that the extended union, the extended intersection, the restricted union and restricted intersection of these ideals

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over a Γ-semigroup is also the same ideal over the same Γ-semigroup.

2. Preliminaries

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-semigroup if it satisfies

(i) $x\gamma y \in S$
(ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

Let A be a non-empty subset of a Γ-semigroup S then A is called

(i) a Γ-subsemigroup of S if $A\Gamma A \subseteq A$.
(ii) a left (right) Γ-ideal of S if $S\Gamma A \subseteq A (A\Gamma S \subseteq A)$.
(iii) a two sided Γ-ideal or simply a Γ-ideal if $S\Gamma A \subseteq A$ and $A\Gamma S \subseteq A$.
(iv) a bi- Γ-ideal of S if A is a Γ-subsemigroup of S and $A\Gamma S\Gamma A \subseteq A$.
(v) an interior Γ-ideal of S if A is Γ-subsemigroup of S and $S\Gamma A\Gamma S \subseteq A$.

A regular element a of a Γ-semigroup S is such that there exists an element $s \in S$ and $\alpha, \beta \in \Gamma$ satisfying $x = x\alpha s\beta x$. S is called a regular Γ-semigroup if every element of S is regular.

Definition 2.1 [12]

A bipolar fuzzy set A in a universe U is an object having the form $A = \left\{ \left\langle x, \mu_A^+(x), \mu_A^-(x) \right\rangle : x \in X \right\}$.

Where $\mu_A^+ : X \to [0,1]$ and $\mu_A^- : X \to [-1,0]$. Here $\mu_A^+(x)$ represents the degree of satisfaction of an element x to the property corresponding to a bipolar fuzzy set $A = \left\{ \left\langle x, \mu_A^+(x), \mu_A^-(x) \right\rangle : x \in X \right\}$ and $\mu_A^-(x)$ represents the degree of satisfaction of x to some implicit counter property of $A = \left\{ \left\langle x, \mu_A^+(x), \mu_A^-(x) \right\rangle : x \in X \right\}$. For the simplicity the symbol $A = \left\langle \mu_A^+, \mu_A^- \right\rangle$ has been used for the bipolar fuzzy set $A = \left\{ \left\langle x, \mu_A^+(x), \mu_A^-(x) \right\rangle : x \in X \right\}$.

Definition 2.2

For any two bipolar fuzzy sets $A = \left\langle \mu_A^+, \mu_A^- \right\rangle$ and $B = \left\langle \mu_B^+, \mu_B^- \right\rangle$ in a universe U,

1. $A \subseteq B$ means that $\mu_A^+(x) \geq \mu_B^+(x)$ and $\mu_A^-(x) \leq \mu_B^-(x)$, for all $x \in X$.
2. $A \cup B = \{(x, \max\{\mu_A^+(x), \mu_B^+(x)\}, \min\{\mu_A^-(x), \mu_B^-(x)\} \mid x \in X)\}$.

$$= \left\langle \mu_A^+(x) \cup \mu_B^+(x), \mu_A^-(x) \cap \mu_B^-(x) \right\rangle$$

3. $A \cap B = \{(x, \min\{\mu_A^+(x), \mu_B^+(x)\}, \max\{\mu_A^-(x), \mu_B^-(x)\} \mid x \in X)\}$.

$$\left\langle \mu_{A\cap B}^+(x), \mu_{A\cap B}^-(x) \right\rangle = \left\langle \mu_A^+(x) \cap \mu_B^+(x), \mu_A^-(x) \cup \mu_B^-(x) \right\rangle.$$

Remark 2.3

From above definitions it is clear that,

(i) $\mu_{A\cup B}^+(x) = \mu_A^+(x) \cup \mu_B^+(x)$
(ii) $\mu_{A\cup B}^-(x) = \mu_A^-(x) \cap \mu_B^-(x)$
(iii) $\mu_{A\cap B}^+(x) = \mu_A^+(x) \cap \mu_B^+(x)$
(iv) $\mu_{A\cap B}^-(x) = \mu_A^-(x) \cup \mu_B^-(x)$

Definition 2.4 [16]

Let U be an initial universe set and E be the set of parameters. Let P(U) denotes the power set of U. Let A be a non-empty subset of E then the pair (F, A) is called a soft set over U, where F is a mapping given by, $F : A \to P(U)$.

Definition 2.5 [15]

Let U be an initial universe set and E be the set of parameters. Let A be a non-empty subset of E and P(FS(U)) be the collection of all fuzzy subsets of U then the pair (f, A) is called a fuzzy soft set over U, where f is a mapping given by, $f : A \to P(FS(U))$.

Definition 2.6 [5]

Let U be an initial universe set and E be the set of parameters. Let $A \subseteq E$ and BF^U denotes the set of all bipolar fuzzy subsets of U. Then a pair (F, A) is called a bipolar fuzzy soft set over U, where F is a mapping given by $F : A \to BF^U$. It is defined as

$$(F, A) = \left\{ \left\langle x, \mu_a^+(x), \mu_a^-(x) \right\rangle : x \in U \text{ and } a \in A \right\}.$$

For any $a \in A$, $F(a) = \left\{ \left\langle x, \mu_{F(a)}^+(x), \mu_{F(a)}^-(x) \right\rangle : x \in U \right\} = \left\langle \mu_{F(a)}^+(x), \mu_{F(a)}^-(x) \right\rangle$.

Definition 2.7 [5]

i) A bipolar fuzzy soft set is called a null bipolar fuzzy soft set if for all
ii) A bipolar fuzzy soft set is called an absolute bipolar fuzzy soft set if for all

DEFINITION 2.8 [5]

The compliment of a bipolar fuzzy soft set (F, A) is denoted by $(F, A)^c$ and is defined as

$$(F,A)^c = \left\{ \left\langle x, 1 - \mu_A^+(x), -1 - \mu_A^-(x) \right\rangle : x \in U \right\}$$

DEFINITION 2.9 [5]

For any two bipolar fuzzy soft sets, (F, A) and (G, B) over a common universe U, we say that (F, A) is a bipolar fuzzy soft subset of (G, B) if $A \subseteq B$ and $F(a) \subseteq G(a)$, for all $a \in A$. We write this as $(F, A) \tilde{\subseteq} (G, B)$

Here (G, B) is called fuzzy soft superset. (F, A) and (G, B) over a common universe U are said to be fuzzy soft equal if, $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$

EXAMPLE 2.10

Let $U = \{x_1, x_2, x_3, x_4\}$ be the universe set and $E = \{a, b, c, d\}$ be the set of parameters. Let $A = \{a, b\} \subseteq E$ and $B = \{a, b, c\}$. Define (F, A) as

$(F, A) = \{F(a), F(b)\}$, where

$F(a) = \{(x_1, 0.1, -0.5), (x_2, 0.2, -0.4), (x_3, 0.3, -0.3)\}$,

$F(b) = \{(x_1, 0.2, -0.5), (x_2, 0.3, -0.2), (x_3, 0.4, -0.4)\}$.

Also

$(G, B) = \{G(a), G(b), G(c)\}$, where

$G(a) = \{(x_1, 0.2, -0.6), (x_2, 0.3, -0.5), (x_3, 0.4, -0.4)\}$

$G(b) = \{(x_1, 0.3, -0.6), (x_2, 0.4, -0.4), (x_3, 0.5, -0.5)\}$,

$G(c) = \{(x_1, 0.2, -0.5), (x_2, 0.3, -0.2), (x_3, 0.6, -0.6)\}$.

Then (F, A) and (G, B) are bipolar fuzzy soft sets over U. Also $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$. Hence $(F, A) \tilde{\subseteq} (G, B)$. Also

$(F, A)^c = \{F(a)^c, F(b)^c\}$, where

$F(a)^c = \{(x_1, 0.9, -0.5), (x_2, 0.8, -0.6), (x_3, 0.7, -0.7)\}$,

$F(b)^c = \{(x_1, 0.8, -0.5), (x_2, 0.7, -0.8), (x_3, 0.6, -0.6)\}$.

DEFINITION 2.11 [5]

Let (F, A) and (G, B) be two bipolar fuzzy soft sets over a common universe U then "(F, A) AND (G, B)", denoted by $(F, A) \tilde{\wedge} (G, B)$ is defined as $(F, A) \tilde{\wedge} (G, B) = (H, C)$ where $C = A \times B$ and $H(a, b) = F(a) \cap F(b)$, for all $(a, b) \in C = A \times B$.

DEFINITION 2.12 [5]

Let (F, A) and (G, B) be two fuzzy soft sets over a common universe U then "(F, A) OR (G, B)", denoted by $(F, A) \tilde{\vee} (G, B)$ is defined as $(F, A) \tilde{\vee} (G, B) = (K, C)$, where $C = A \times B$ and $K(a, b) = F(a) \cup G(b)$, for all $(a, b) \in C = A \times B$.

DEFINITION 2.13

Let (F, A) and (G, B) be two bipolar fuzzy soft sets over a common universe U then their "extended union" is a bipolar fuzzy soft set over U denoted by $(F, A) \cup_E (G, B)$ and is defined as $(F, A) \cup_E (G, B) = (H, C)$, where $C = A \cup B$ and $H : C \to BF^U$ is given as,

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \quad \text{for all } c \in C. \\ \max\{F(c), G(c)\} & \text{if } c \in A \cap B \end{cases}$$

DEFINITION 2.14

Let (F, A) and (G, B) be two bipolar fuzzy soft sets over a common universe U then their "extended intersection" is a bipolar fuzzy soft set over U denoted by $(F, A) \cap_E (G, B)$ and is defined as $(F, A) \cap_E (G, B) = (H, C)$, where $C = A \cup B$ and $H : C \to BF^U$ is given as,

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \quad \text{for all } c \in C. \\ \min\{F(c), G(c)\} & \text{if } c \in A \cap B \end{cases}$$

DEFINITION 2.15

Let (F, A) and (G, B) be two bipolar fuzzy soft sets over a common universe U such that $A \cap B \neq \phi$. The "restricted union" of (F, A) and (G, B) is defined to be a bipolar fuzzy soft set (H, C) over U where $C = A \cap B$ and for $H(c) = F(c) \cup G(c)$, for all $c \in C$. This is denoted by $(H, C) = (F, A) \cup_R (G, B)$.

DEFINITION 2.16

Let (F, A) and (G, B) be two bipolar fuzzy soft sets over a common universe U such that $A \cap B \neq \phi$. The "restricted intersection" of (F, A) and (G, B) is defined to be a bipolar fuzzy soft set (H, C) over U where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$, for all $c \in C$. This is denoted by $(H, C) = (F, A) \cup_R (G, B)$.

If $\{(F_i, A_i): i \in I\}$ be a collection of bipolar fuzzy soft sets over a common universe U such that $\underset{i \in I}{\cap} A_i \neq \phi$, then similarly, we can define $\underset{i \in I}{\cup_R}(F_i, A_i)$ and $\underset{i \in I}{\cap_R}(F_i, A_i)$.

3. Main Section

In this section S will denotes a Γ-semigroup otherwise it will be specified.

DEFINITION 3.1

A bipolar fuzzy soft set (F, A) over a Γ-semigroup S is called a bipolar fuzzy soft Γ-subsemigroup over S if

$$\mu^+_{F(a)}(x\gamma y) \geq \min\{\mu^+_{F(a)}(x), \mu^+_{F(a)}(y)\},$$

$$\text{and } \mu^-_{F(a)}(x\gamma y) \leq \max\{\mu^-_{F(a)}(x), \mu^-_{F(a)}(y)\},$$

for all $x, y \in S, \gamma \in \Gamma$ and $a \in A$.

DEFINITION 3.2

A bipolar fuzzy soft set (F, A) over a Γ-semigroup S is called a bipolar fuzzy soft left (right) Γ-ideal over S if

$$\text{and } \mu^+_{F(a)}(x\gamma y) \geq \mu^+_{F(a)}(y) \ \left(\mu^+_{F(a)}(xzy) \geq \mu^+_{F(a)}(x)\right)$$

$$\mu^-_{F(a)}(xzy) \leq \mu^-_{F(a)}(y) \ \left(\mu^-_{F(a)}(xzy) \leq \mu^-_{F(a)}(x)\right),$$

for all $x, y \in S, \gamma \in \Gamma$ and $a \in A$.

DEFINITION 3.3

A bipolar fuzzy soft set (F, A) over a Γ-semigroup S is called a bipolar fuzzy soft Γ-ideal over S if it is a bipolar fuzzy soft left Γ-ideal and a bipolar fuzzy soft right Γ-ideal over S.

REMARK 3.4

(i) *Every bipolar fuzzy soft subset over a Γ-semigroup S may not be a bipolar fuzzy soft Γ-subsemigroup over S.*
(ii) *Every bipolar fuzzy soft left (right) Γ-ideal over a Γ-semigroup S is a bipolar fuzzy soft Γ-subsemigroup over S but the converse is not true.*

EXAMPLE 3.5

Let $S = \{x_1, x_2, x_3\}$ and $\Gamma = \{\gamma\}$, then S is a Γ-semigroup with the following table,

γ	x_1	x_2	x_3
x_1	x_1	x_3	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

Let $E = \{a, b, c\}, A = \{a, c\}$ then (F, A) is a bipolar fuzzy soft set defined as, $(F, A) = \{F(a), F(c)\}$, where $F(a) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.3), (x_3, 0.7, -0.4)\}$ and $F(c) = \{(x_1, 0.3, -0.3), (x_2, 0.4, -0.5), (x_3, 0.5, -0.7)\}$. It is easy to verify that (F, A) is a bipolar fuzzy soft left Γ-ideal and a bipolar fuzzy soft right Γ-ideal over S. Hence (F, A) is a bipolar fuzzy soft Γ-ideal over S.

Let $B = \{a, b\}$ and $(G, B) = \{G(a), G(b)\}$, where

$G(a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.3, -0.3)\}$ and
$G(b) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\}$ then
(G, B) is a fuzzy soft Γ-subsemigroup but it is neither a bipolar fuzzy soft left Γ-ideal nor a bipolar fuzzy soft right Γ-ideal over S. As we can see below,

for left Γ-ideal,

$$\mu^+_{G(a)}(x_1\gamma x_2) = \mu^+_{G(a)}(x_3) = 0.3 \not\geq 0.6 = \mu^+_{G(a)}(x_2).$$

Also for right Γ-ideal,

$$\mu^+_{G(a)}(x_2\gamma x_1) = \mu^+_{G(a)}(x_3) = 0.3 \not\geq 0.6 = \mu^+_{G(a)}(x_2).$$

Let $C = \{b, c\} \subseteq E$, and $(H, C) = \{H(b), H(c)\}$, where
$H(b) = \{(x_1, 0.5, -0.8), (x_2, 0.8, -0.5), (x_3, 0.1, -0.4)\}$ and
$H(c) = \{(x_1, 0.7, -0.5), (x_2, 0.6, -0.7), (x_3, 0.5, -0.2)\}$. Then
(H, C) is a bipolar fuzzy soft set over S but it is not a Γ-subsemigroup over S, as

$$\mu^+_{H(b)}(x_1\gamma x_2) = \mu^+_{H(b)}(x_3) = 0.1 \not\geq 0.5$$

$$= \min\{\mu^+_{H(b)}(x_1), \mu^+_{H(b)}(x_2)\}.$$

DEFINITION 3.6

A bipolar fuzzy soft Γ-semigroup (F, A) of S is called a bipolar fuzzy soft interior Γ-ideal over S if

$$\mu^+_{F(a)}(x\alpha z\beta y) \geq \mu^+_{F(a)}(z) \text{ and } \mu^-_{F(a)}(x\alpha z\beta y) \leq \mu^-_{F(a)}(z),$$

for all $x, y, z \in S, \alpha, \beta \in \Gamma$ and $a \in A$.

REMARK 3.7

Every bipolar fuzzy soft Γ-ideal over a Γ-semigroup S is a bipolar fuzzy interior Γ-ideal over S but the converse is not true.

Example 3.8

Let $S = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\alpha, \beta\}$, then S is a Γ-semigroup with the following table,

$$
\begin{array}{c|cccc}
\alpha & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_1 & x_1 & x_1 \\
x_2 & x_1 & x_3 & x_1 & x_2 \\
x_3 & x_1 & x_3 & x_1 & x_4 \\
x_4 & x_1 & x_1 & x_1 & x_3
\end{array}
\quad and \quad
\begin{array}{c|cccc}
\beta & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_1 & x_1 & x_1 \\
x_2 & x_2 & x_2 & x_2 & x_2 \\
x_3 & x_1 & x_1 & x_1 & x_1 \\
x_4 & x_1 & x_1 & x_1 & x_4
\end{array}
$$

Let $E = \{a, b, c, d\}$ and $A = \{a, b\}$. Define the bipolar fuzzy soft set (F, A) as, $(F, A) = \{F(a), F(b)\}$, where

$$F(a) = \{(x_1, 0.9, -0.7), (x_2, 0.7, -0.5),$$
$$(x_3, 0.6, -0.3), (x_4, 0.1, -0.1)\}$$
$$and$$
$$F(b) = \{(x_1, 0.6, -0.8), (x_2, 0.4, -0.6),$$
$$(x_3, 0.3, -0.4), (x_4, 0.1, -0.2)\}.$$

Then (F, A) is a bipolar fuzzy soft interior Γ-ideal over S but it is not a Γ-ideal over S as we can see bellow

$$\mu^+_{F(a)}(x_3 \alpha x_4 \beta x_4) = \mu^+_{F(a)}(x_4 \beta x_4) = \mu^+_{F(a)}(x_4)$$
$$= 0.1 \not\geq 0.6 = \mu^+_{F(a)}(x_3),$$

so (F, A) is not a bipolar fuzzy soft right Γ-ideal over S and hence not a bipolar fuzzy soft Γ-ideal over S.

Definition 3.9

A bipolar fuzzy soft Γ-subsemigroup (F, A) of S is called a bipolar fuzzy soft bi-Γ-ideal over S if

$$\mu^+_{F(a)}(x \alpha z \beta y) \geq \min\{\mu^+_{F(a)}(x), \mu^+_{F(a)}(y)\},$$
$$and\ \mu^-_{F(a)}(x \alpha z \beta y) \leq \max\{\mu^-_{F(a)}(x), \mu^-_{F(a)}(y)\},$$

for all $x, y, z \in S$, $\alpha, \beta \in \Gamma$ and $a \in A$.

Example 3.10

Let $S = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ then S is a Γ-semigroup with the following table,

$$
\begin{array}{c|cccc}
\alpha & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_1 & x_1 & x_1 \\
x_2 & x_2 & x_2 & x_2 & x_2 \\
x_3 & x_1 & x_1 & x_1 & x_3 \\
x_4 & x_1 & x_1 & x_1 & x_4
\end{array}
\quad and \quad
\begin{array}{c|cccc}
\beta & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_1 & x_1 & x_1 \\
x_2 & x_1 & x_2 & x_1 & x_1 \\
x_3 & x_1 & x_1 & x_3 & x_1 \\
x_4 & x_1 & x_1 & x_1 & x_4
\end{array}
\quad and \quad
\begin{array}{c|cccc}
\gamma & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_1 & x_1 & x_1 \\
x_2 & x_1 & x_3 & x_1 & x_2 \\
x_3 & x_1 & x_3 & x_1 & x_4 \\
x_4 & x_1 & x_1 & x_1 & x_3
\end{array}
$$

Let $E = \{a, b, c, d\}$ and $A = \{a, b\}$. Define the bipolar fuzzy soft set (F, A) as, $(F, A) = \{F(a), F(b)\}$, where

$$F(a) = \{(x_1, 0.8, -0.9), (x_2, 0.6, -0.7),$$
$$(x_3, 0.4, -0.6), (x_4, 0.2, -0.3)\}$$
$$and$$
$$F(b) = \{(x_1, 0.7, -0.6), (x_2, 0.5, -0.4),$$
$$(x_3, 0.3, -0.2), (x_4, 0.1, -0.1)\}.$$

Then (F, A) is a bipolar fuzzy soft bi-Γ-ideal over S.

Lemma 3.11

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a Γ-semigroup S then $(F, A) \,\tilde{\wedge}\, (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

Proof:

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups over a Γ-semigroup S, then as defined $(F, A) \,\tilde{\wedge}\, (G, B)$ where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in C = A \times B$. As (F, A) and (G, B) are bipolar fuzzy soft Γ-subsemigroups over S then for $(a, b) \in C = A \times B$ and $x, y \in S$, $\gamma \in \Gamma$,

$$\mu^+_{H(a,b)}(x \gamma y)$$
$$= (\mu^+_{F(a)} \cap \mu^+_{G(b)})(x \gamma y) = \min\{\mu^+_{F(a)}(x \gamma y), \mu^+_{G(b)}(x \gamma y)\}$$
$$\geq \min\{\min\{\mu^+_{F(a)}(x), \mu^+_{F(a)}(y)\}, \min\{\mu^+_{G(b)}(x), \mu^+_{G(b)}(y)\}\}$$
$$= \min\{\min\{\mu^+_{F(a)}(x), \mu^+_{G(b)}(x)\}, \min\{\mu^+_{F(a)}(y), \mu^+_{G(b)}(y)\}$$
$$= \min\{(\mu^+_{F(a)} \cap \mu^+_{G(b)})(x), (\mu^+_{F(a)} \cap \mu^+_{G(b)})(y)\}$$
$$= \min\{\mu^+_{H(a,b)}(x), \mu^+_{H(a,b)}(y)\}.$$

And
$$\mu^-_{H(a,b)}(x \gamma y)$$
$$= (\mu^-_{F(a)} \cup \mu^-_{G(b)})(x \gamma y) = \max\{\mu^-_{F(a)}(x \gamma y), \mu^-_{G(b)}(x \gamma y)\}$$
$$\leq \max\{\max\{\mu^-_{F(a)}(x), \mu^-_{F(a)}(y)\}, \max\{\mu^-_{G(b)}(x), \mu^-_{G(b)}(y)\}\}$$
$$= \max\{\max\{\mu^-_{F(a)}(x), \mu^-_{G(b)}(x)\}, \max\{\mu^-_{F(a)}(y), \mu^-_{G(b)}(y)\}$$
$$= \max\{(\mu^-_{F(a)} \cup \mu^-_{G(b)})(x), (\mu^-_{F(a)} \cup \mu^-_{G(b)})(y)\}$$
$$= \max\{\mu^-_{H(a,b)}(x), \mu^-_{H(a,b)}(y)\}$$

Hence, $(H, C) = (F, A) \,\tilde{\wedge}\, (G, B)$ is a bipolar fuzzy soft Γ-subsemigroup over S. The other cases can be proved in a similar way.

Lemma 3.12

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a

Γ-semigroup S then $(F, A) \tilde{\vee} (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

PROOF:

Straightforward.

EXAMPLE 3.13

Let $S = \{x_1, x_2, x_3\}$ and $\Gamma = \{\gamma\}$ be the Γ-semigroup along with the table given in Example 3.5. Let $E = \{a, b, c, d\}$ and $A = \{a, b\} \subseteq E$ then (F, A) is a bipolar fuzzy soft set defined as, $(F, A) = \{F(a), F(b)\}$,
where

$$F(a) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.5), (x_3, 0.3, -0.4)\}$$
$$F(b) = \{(x_1, 0.3, -0.3), (x_2, 0.8, -0.6), (x_3, 0.5, -0.5)\}.$$

Let $B = \{a, c\} \subseteq E$ then $(G, B) = \{G(a), G(c)\}$,
where

$$G(a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.4, -0.3)\}$$
$$G(c) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\}.$$

Then (F, A) and (G, B) are bipolar fuzzy soft Γ-subsemigroups of S.
Now $(F, A) \tilde{\wedge} (G, B) = (H, C)$, where, $C = A \times B = \{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$ and $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in A \times B = C$.
Then $(H, C) = \{H(a, a), H(a, c), H(b, a), H(b, c)\}$,
where

$$H(a, a) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.4), (x_3, 0.3, -0.3)\},$$
$$H(a, c) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.5), (x_3, 0.3, -0.4)\},$$
$$H(b, a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.4, -0.3)\},$$
$$H(b, c) = \{(x_1, 0.3, -0.3), (x_2, 0.7, -0.6), (x_3, 0.5, -0.4)\}.$$

Obviously, for all $x, y \in S, \gamma \in \Gamma$ and $(a, b) \in C$,

$$\mu^+_{H(a,b)}(x \gamma y) \geq \min\{\mu^+_{H(a,b)}(x), \mu^+_{H(a,b)}(y)\}$$

and $\mu^-_{H(a,b)}(x \gamma y) \leq \max\{\mu^-_{H(a,b)}(x), \mu^-_{H(a,b)}(y)\}$.

Hence, $(H, C) = (F, A) \tilde{\wedge} (G, B)$ is a bipolar fuzzy soft Γ-subsemigroup over S. Now as $(F, A) \tilde{\vee} (G, B) = (H, C)$, where, $C = A \times B = \{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$ and $H(a, b) = F(a) \cup G(b)$, for all $(a, b) \in A \times B = C$.
Then $(H, C) = \{H(a, a), H(a, c), H(b, a), H(b, c)\}$, where

$$H(a, a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.5), (x_3, 0.4, -0.4)\},$$
$$H(a, c) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\},$$

$$H(b, a) = \{(x_1, 0.3, -0.3), (x_2, 0.8, -0.6), (x_3, 0.5, -0.5)\},$$
$$H(b, c) = \{(x_1, 0.4, -0.3), (x_2, 0.8, -0.8), (x_3, 0.5, -0.5)\}.$$

Obviously, $(H, C) = (F, A) \tilde{\vee} (G, B)$ is a bipolar fuzzy soft Γ-subsemigroup over S. Similarly, we can construct examples for left (right, interior and bi) Γ-ideals.

LEMMA 3.14

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a Γ-semigroup S then $(F, A) \cap_R (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

PROOF:

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups over a Γ-semigroup S then $(F, A) \cap_R (G, B) = (H, C)$, where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. Now as,

$$\mu^+_{H(c)}(x \gamma y) = \mu^+_{F(c) \cap G(c)}(x \gamma y) = \mu^+_{F(c)}(x \gamma y) \cap \mu^+_{G(c)}(x \gamma y)$$

$$= \min\{\mu^+_{F(c)}(x \gamma y), \mu^+_{G(c)}(x \gamma y)\}$$

$$\geq \min\{\min\{\mu^+_{F(c)}(x), \mu^+_{F(c)}(y)\}, \min\{\mu^+_{G(c)}(x), \mu^+_{G(c)}(y)\}\}$$

$$= \min\{\min\{\mu^+_{F(c)}(x), \mu^+_{G(c)}(x)\}, \min\{\mu^+_{F(c)}(y), \mu^+_{G(c)}(y)\}\}$$

$$= \min\{\mu^+_{F(c)}(x) \cap \mu^+_{G(c)}(x), \mu^+_{F(c)}(y) \cap \mu^+_{G(c)}(y)\}$$

$$= \min\{\mu^+_{H(c)}(x), \mu^+_{H(c)}(y)\}.$$

Similarly, $\mu^-_{H(c)}(x \gamma y) \leq \max\{\mu^-_{H(c)}(x), \mu^-_{H(c)}(y)\}$
Hence, $(H, C) = (F, A) \cap_R (G, B)$ is a bipolar fuzzy soft Γ-subsemigroup over S. The other cases we can prove in same way.

LEMMA 3.15

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a Γ-semigroup S then $(F, A) \cup_R (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

PROOF:

Straightforward.

LEMMA 3.16

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a Γ-semigroup S then $(F, A) \cap_E (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

PROOF:

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups over a Γ-semigroup S then $(F,A) \cap_E (G,B) = (H,C)$ where $C = A \cup B$ and for

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \quad \text{for all } c \in C. \\ \min\{F(c), G(c)\} & \text{if } c \in A \cap B. \end{cases}$$

Let $c \in C$, $x, y \in S$ and $\gamma \in \Gamma$, we have the following cases,

(i) $c \in A - B$ implies that $H(C) = F(C)$ We have

$$\mu^+_{H(c)}(x\gamma y) = \mu^+_{F(c)}(x\gamma y) \geq \min\{\mu^+_{F(c)}(x), \mu^+_{F(c)}(y)\}$$
$$= \min\{\mu^+_{H(c)}(x), \mu^+_{H(c)}(y)\}$$

Also
$$\mu^-_{H(c)}(x\gamma y) = \mu^-_{F(c)}(x\gamma y) \leq \max\{\mu^-_{F(c)}(x), \mu^-_{F(c)}(y)\}$$
$$= \max\{\mu^-_{H(c)}(x), \mu^-_{H(c)}(y)\}$$

(ii) $c \in B - A$ is same as (i).

(iii) $c \in A \cap B$ implies that $H(c) = \min\{F(c), G(c)\} = F(c) \cap G(c)$.

Then as proved in above lemma

$$\mu^+_{H(c)}(x\gamma y) \geq \min\{\mu^+_{H(c)}(x), \mu^+_{H(c)}(y)\}$$

and $\quad \mu^-_{H(c)}(x\gamma y) \leq \max\{\mu^-_{H(c)}(x), \mu^-_{H(c)}(y)\}$.

Hence $(F, A) \cap_E (G, B)$ is a bipolar fuzzy soft Γ-subsemigroup over S. The other cases can be proved in a similar way.

LEMMA 3.17

Let (F, A) and (G, B) be two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals) over a Γ-semigroup S then $(F,A) \cup_E (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup (left Γ-ideal, right Γ-ideal) over S.

PROOF: Straightforward.

EXAMPLE 3.18

Let $S = \{x_1, x_2, x_3\}$ and $\Gamma = \{\gamma\}$ be the Γ-semigroup along with the table given in Example 3.5. Let $E = \{a,b,c,d\}$ and $A = \{a,b\}$, $B = \{a,b,c\}$, then (F, A) and (G, B) are a bipolar fuzzy soft sets defined as, $(F,A) = \{F(a), F(b)\}$, where

$$F(a) = \{(x_1, 0.1, -0.2), (x_2, 0.7, -0.3), (x_3, 0.8, -0.4)\}$$
$$F(b) = \{(x_1, 0.3, -0.3), (x_2, 0.5, -0.4), (x_3, 0.6, -0.5)\},$$

and $(G, B) = \{G(a), G(b), G(c)\}$, where

$$G(a) = \{(x_1, 0.3, -0.4), (x_2, 0.4, -0.6), (x_3, 0.7, -0.6)\}$$
$$G(b) = \{(x_1, 0.2, -0.1), (x_2, 0.6, -0.2), (x_3, 0.8, -0.3)\}$$
$$G(c) = \{(x_1, 0.4, -0.5), (x_2, 0.7, -0.7), (x_3, 0.9, -0.8)\}.$$

Obviously, (F, A) and (G, B) are bipolar fuzzy soft left Γ-ideals over S. As $(F, A) \cap_R (G, B) = (H,C)$, where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. Since, $C = A \cap B = \{a,b\}$ then $(H,C) = \{H(a), H(b)\}$, where

$$H(a) = \{(x_1, 0.1, -0.2), (x_2, 0.4, -0.3), (x_3, 0.7, -0.4)\}$$
$$H(b) = \{(x_1, 0.2, -0.1), (x_2, 0.4, -0.2), (x_3, 0.6, -0.3)\}$$

It is easy to verify that

$$\mu^+_{H(c)}(x\gamma y) \geq \mu^+_{H(c)}(y)$$
$$\text{and} \quad \mu^-_{H(c)}(x\gamma y) \leq \mu^-_{H(c)}(y)$$

Also $(F, A) \cup_R (G, B) = (H', C)$, where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$. Since $C = A \cap B = \{a,b\}$ then $(H', C) = \{H'(a), H'(b)\}$ where,

$$H'(a) = \{(x_1, 0.3, -0.4), (x_2, 0.7, -0.6), (x_3, 0.8, -0.6)\}$$
$$H'(b) = \{(x_1, 0.3, -0.3), (x_2, 0.6, -0.4), (x_3, 0.8, -0.5)\}.$$

Easily we can show that

$$\mu^+_{H'(c)}(x\gamma y) \geq \mu^+_{H'(c)}(y)$$
$$\text{and} \quad \mu^-_{H'(c)}(x\gamma y) \leq \mu^-_{H'(c)}(y).$$

Hence, $(F, A) \cap_R (G, B)$ and $(F, A) \cap_R (G, B)$ are bipolar fuzzy soft left Γ-ideals over S. Similarly, we can construct examples for $(F, A) \cap_E (G, B)$ and $(F, A) \cap_E (G, B)$.

LEMMA 3.19

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S. then $(F, A) \tilde{\wedge} (G, B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

PROOF:

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals over a Γ-semigroup S. then they are also bipolar fuzzy soft Γ-subsemigroups and by Lemma 3.11, $(F, A) \tilde{\wedge} (G, B)$ is also a bipolar fuzzy soft Γ-subsemigroup. Since, $(F, A) \tilde{\wedge} (G, B) = (H, C)$, where $C = A \times B$ and $H(a,b) = F(a) \cap G(b)$.
Let $x, y, z \in S$, $\alpha, \beta \in \Gamma$ and $(a,b) \in C = A \times B$, then

$$\mu^+_{H(a,b)}(x\alpha z\beta y)$$

$$= (\mu^+_{F(a)\cap G(b)})(x\alpha z\beta y) = (\mu^+_{F(a)})(x\alpha z\beta y)\cap(\mu^+_{G(b)})(x\alpha z\beta y)$$

$$= \min\{\mu^+_{F(a)}(x\alpha z\beta y),\mu^+_{G(b)}(x\alpha z\beta y)\}$$

$$\geq \min\{\min\{\mu^+_{F(a)}(x),\mu^+_{F(a)}(y)\},\min\{\mu^+_{G(b)}(x),\mu^+_{G(b)}(y)\}\}$$

$$= \min\{\min\{\mu^+_{F(a)}(x),\mu^+_{G(b)}(x)\},\min\{\mu^+_{F(a)}(y),\mu^+_{G(b)}(y)\}\}$$

$$= \min\{(\mu^+_{F(a)}\cap\mu^+_{G(b)})(x),(\mu^+_{F(a)}\cap\mu^+_{G(b)})(y)\}$$

$$= \min\{\mu^+_{H(a,b)}(x),\mu^+_{H(a,b)}(y)\}.$$

And

$$\mu^-_{H(a,b)}(x\alpha z\beta y)$$

$$= (\mu^+_{F(a)\cap G(b)})(x\alpha z\beta y) = \mu^-_{F(a)}(x\alpha z\beta y)\cup\mu^-_{G(b)}(x\alpha z\beta y)$$

$$= \max\{\mu^-_{F(a)}(x\alpha z\beta y),\mu^-_{G(b)}(x\alpha z\beta y)\}$$

$$\leq \max\{\max\{\mu^-_{F(a)}(x),\mu^-_{F(a)}(y)\},\max\{\mu^-_{G(b)}(x),\mu^-_{G(b)}(y)\}\}$$

$$= \max\{\max\{\mu^-_{F(a)}(x),\mu^-_{G(b)}(x)\},\max\{\mu^-_{F(a)}(y),\mu^-_{G(b)}(y)\}\}$$

$$= \max\{(\mu^-_{F(a)}\cup\mu^-_{G(b)})(x),(\mu^-_{F(a)}\cup\mu^-_{G(b)})(y)\}$$

$$= \max\{\mu^-_{H(a,b)}(x),\mu^-_{H(a,b)}(y)\}.$$

Hence $(H,C)=(F,A)\tilde{\wedge}(G,B)$ is a bipolar fuzzy soft bi-Γ-ideal over S. The other case can be proved in a similar way.

Lemma 3.20

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S then $(F,A)\tilde{\vee}(G,B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

Proof:

Straightforward.

Lemma 3.21

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S then $(F, A)\cap_R(G,B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

Proof:

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals over Γ-semigroup S then (F, A) and (G, B) are bipolar fuzzy soft Γ-subsemigroups over S. By Lemma 3.14,

$(F,A)\cap_R(G,B)$ is also a bipolar fuzzy soft Γ-subsemigroup over S.

Since, $(F, A) \cap_R(G,B)=(H,C)$, where $C = A\cap B$ and $H(c) = F(c)\cap G(c)$, for all $c\in C$. Then for all $x, y, z\in S$, $\alpha,\beta\in\Gamma$ and $c\in C$,

$$\mu^+_{H(c)}(x\alpha z\beta y)$$

$$= \mu^+_{F(c)\cap G(c)}(x\alpha z\beta y) = \mu^+_{F(c)}(x\alpha z\beta y)\cap\mu^+_{G(c)}(x\alpha z\beta y)$$

$$= \min\{\mu^+_{F(c)}(x\alpha z\beta y),\mu^+_{G(c)}(x\alpha z\beta y)\}$$

$$\geq \min\{\min\{\mu^+_{F(c)}(x),\mu^+_{F(c)}(y)\},\min\{\mu^+_{G(c)}(x),\mu^+_{G(c)}(y)\}\}$$

$$= \min\{\min\{\mu^+_{F(c)}(x),\mu^+_{G(c)}(x)\},\min\{\mu^+_{F(c)}(y),\mu^+_{G(c)}(y)\}\}$$

$$= \min\{\mu^+_{F(c)}(x)\cap\mu^+_{G(c)}(x),\mu^+_{F(c)}(y)\cap\mu^+_{G(c)}(y)\}$$

$$= \min\{\mu^+_{H(c)}(x),\mu^+_{H(c)}(y)\}.$$

Similarly, $\mu^-_{H(c)}(x\gamma y)\leq\max\{\mu^-_{H(c)}(x),\mu^-_{H(c)}(y)\}$. Hence, $(H,C)=(F,A)\cap_R(G,B)$ is a bipolar fuzzy soft bi-Γ-ideal over S. The case for interior Γ-ideals can be proved in similar way.

Lemma 3.22

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S then $(F, A)\cup_R(G,B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

Proof:

Straightforward.

Lemma 3.23

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S then $(F, A)\cap_E(G,B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

Proof:

Straightforward.

Lemma 3.24

Let (F, A) and (G, B) be two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semigroup S then $(F,A)\cup_E(G,B)$ is also a bipolar fuzzy soft bi-Γ-ideal (interior Γ-ideal) over S.

Proof:

Straightforward.

COROLLARY 3.25

The restricted union $\cup_R$ and restricted intersection $\cap_R$ of two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals, bi-Γ-ideals, interior Γ-ideals) over a Γ-semigroup S is also a bipolar fuzzy Γ-subsemigroup(left Γ-ideal, right Γ-ideal, bi-Γ-ideal, interior Γ-ideal) over S.

COROLLARY 3.26

The extended union $\cup_E$ and extended intersection $\cap_E$ of two bipolar fuzzy soft Γ-subsemigroups (left Γ-ideals, right Γ-ideals, bi-Γ-ideals, interior Γ-ideals) over a Γ-semigroup S is also a bipolar fuzzy Γ-subsemigroup(left Γ-ideal, right Γ-ideal, bi-Γ-ideal, interior Γ-ideal) over S.

THEOREM 3.27

Let T be a parameter set and $\Sigma_T(S)$ be the set of all bipolar fuzzy soft left Γ-ideals over S. Then $(\Sigma_T(S), \cup_E, \cap_R)$ forms a complete distributive lattice along with the relation $\tilde{\subseteq}$.

PROOF:

Consider two bipolar fuzzy soft left Γ-ideals (F, A) and (G, B) over S with $A \subseteq T$ and $B \subseteq T$. i.e, $(F, A), (G, B) \in \Sigma_T(S)$ then, $(F, A) \cap_R (G, B), (F, A) \cup_E (G, B) \in \Sigma_T(S)$ by Lemma 3.14 and Lemma 3.17. Obviously, $(F, A) \cap_R (G, B)$ is the greatest lower of $\{(F, A), (G, B)\}$ and $(F, A) \cup_E (G, B)$ is the least upper bound of $\{(F, A), (G, B)\}$. So every sub-collection of has a least upper bound as well as a greatest lower bound. Hence $\Sigma_T(S)$ is a complete lattice. Now let, $(F, A), (G, B), (H, C) \in \Sigma_T(S)$, then

$(F, A) \cap_R ((G, B) \cup_E (H, C)) = (I, A \cap (B \cup C))$. Also

$$((F, A) \cap_R (G, B)) \cup_E ((F, A) \cap_R (H, C))$$
$$= (J, (A \cap B) \cup (A \cap C))$$
$$= (J, A \cap (B \cup C)).$$

Clearly, for $x \in A \cap (B \cup C), I(x) = J(x)$. Hence,

$$(F, A) \cap_R ((G, B) \cup_E (H, C))$$
$$= ((F, A) \cap_R (G, B)) \cup_E ((F, A) \cap_R (H, C)).$$

Which implies that $\Sigma_T(S)$ forms a complete distributive lattice over S.

Note: Similarly, we can prove the above result for right Γ-ideals, bi Γ-ideals and interior Γ-ideals.

THEOREM 3.28

Let T be a parameter set and $\Sigma_T(S)$ be the set of all bipolar fuzzy soft left Γ-ideals over S. Then $(\Sigma_T(S), \cup_R, \cap_E)$ forms a complete distributive lattice along with the relation $\tilde{\subseteq}'$.

PROOF:

Similar as above theorem.

Let T be a parameter set and $D \subseteq T$. If $\Sigma_D(S)$ be the collection of all bipolar fuzzy soft left (right, bi, interior) Γ-ideals S with parameter set D. Then for $D_1, D_2 \subseteq D$. we have the following results.

LEMMA 3.29

If (F, D_1) and $(G, D_2) \in \Sigma_D(S)$, then $(F, D_1) \cap_R (G, D_2) \in \Sigma_D(S)$ and $(F, A) \cup_R (G, B) \in \Sigma_D(S)$.

PROOF.

Straightforward.

LEMMA 3.30

If (F, D_1) and $(G, D_2) \in \Sigma_D(S)$, then $(F, D_1) \cap_E (G, D_2) \in \Sigma_D(S)$ and $(F, A) \cup_E (G, B) \in \Sigma_D(S)$.

PROOF.

Straightforward.

From above we can write the following remarks

REMARK 3.31

The collection $(\Sigma_D(S), \cup_E, \cap_R)$ forms a sublattice of the collection $(\Sigma_T(S), \cup_E, \cap_R)$.

REMARK 3.32

The collection $(\Sigma_D(S), \cup_R, \cap_E)$ forms a sublattice of the collection $(\Sigma_T(S), \cup_R, \cap_E)$.

DEFINITION 3.33

Let S be a Γ-semigroup and $(F, A), (G, B)$ be two bipolar fuzzy soft sets over S. We define the product of (F, A) and (G, B) as $(F, A) \circ_\Gamma (G, B) = (F\Gamma G, C), where\ C = A \cup B,$

$$\mu_{(F\Gamma G)(c)}^+(s) = \begin{cases} \mu_{F(c)}^+(s) & \text{if } c \in A - B \\ \mu_{G(c)}^+(s) & \text{if } c \in B - A \\ \sup_{s=s_1\gamma s_2} \min\{\mu_{F(z)}^+(s_1), \mu_{G(z)}^+(s_2)\} & \text{if } c \in A \cap B \end{cases}$$

and

$$\mu^-_{(F\Gamma G)(c)}(s) = \begin{cases} \mu^-_{F(c)}(s) & \text{if } c \in A - B \\ \mu^-_{G(c)}(s) & \text{if } c \in B - A \\ \inf\limits_{s=s_1\gamma s_2} \max\{\mu^-_{F(c)}(s_1), \mu^-_{G(c)}(s_2)\} & \text{if } c \in A \cap B \end{cases}$$

for all $c \in C$ *and* $s, s_1, s_2 \in S$ *and* $\gamma \in \Gamma$.

THEOREM 3.34

For any two bipolar fuzzy soft left (right, bi, interior) Γ-ideals (F, A) *and* (G, B) *over S their product* $(F, A) \circ_\Gamma (G, B)$ *is also a bipolar fuzzy soft left (right, bi, interior) Γ-ideal over the Γ-semigroup S.*

PROOF:

For any $c \in C$ we have the following cases,

(i) If $c \in A - B$ then for $x, y \in S$ and $\gamma \in \Gamma$,

$$\mu^+_{(F\Gamma G)(c)}(x\gamma y) = \mu^+_{(F)(c)}(x\gamma y) \geq \mu^+_{(F)(c)}(y)$$
$$= \mu^+_{(F\Gamma G)(c)}(y).$$

Also

$$\mu^-_{(F\Gamma G)(c)}(x\gamma y) = \mu^-_{(F)(c)}(x\gamma y) \leq \mu^-_{(F)(c)}(y)$$
$$= \mu^-_{(F\Gamma G)(c)}(y).$$

(ii) If $c \in A - B$ then same as (i).

(iii) If $c \in A - B$ then for $x, y \in S$ and $\gamma \in \Gamma$,

$$\mu^+_{(F\Gamma G)(c)}(y) = \sup_{y=m\beta n} \{\min\{\mu^+_{F(c)}(m), \mu^+_{G(c)}(n)\}\}$$
$$\leq \sup_{x\gamma y=x\gamma m\beta n} \{\min\{\mu^+_{F(z)}(x\gamma m), \mu^+_{G(z)}(n)\}\}$$
$$= \sup_{x\gamma y=m_1\beta n} \{\min\{\mu^+_{F(z)}(m_1), \mu^+_{G(z)}(n)\}\}$$
$$= \mu^+_{(F\Gamma G)(c)}(x\gamma y).$$

Also,

$$\mu^-_{(F\Gamma G)(c)}(y) = \inf_{y=m\beta n} \{\max\{\mu^-_{F(c)}(m), \mu^-_{G(c)}(n)\}\}$$
$$\geq \inf_{x\gamma y=x\gamma m\beta n} \{\max\{\mu^{-+}_{F(z)}(x\gamma m), \mu^-_{G(z)}(n)\}\}$$
$$= \inf_{x\gamma y=m_1\beta n} \{\max\{\mu^-_{F(z)}(m_1), \mu^-_{G(z)}(n)\}\}$$
$$= \mu^-_{(F\Gamma G)(c)}(x\gamma y).$$

In all cases, we have

$$\mu^+_{(F\Gamma G)(c)}(x\gamma y) \geq \mu^+_{(F\Gamma G)(c)}(y) \text{ and}$$
$$\mu^-_{(F\Gamma G)(c)}(x\gamma y) \leq \mu^-_{(F\Gamma G)(c)}(y).$$

This implies that, $(F, A) \circ_\Gamma (G, B)$ is a bipolar fuzzy soft left Γ-ideal over S.

Note: Similarly, we can prove the above result for bipolar fuzzy soft right Γ-ideals, bi-Γ-ideals and interior Γ-ideals.

THEOREM 3.35

Consider a Γ-semigroup S with identity e. Let $\Delta_T(S)$ *denotes the set of all bipolar fuzzy soft left Γ-ideals (right Γ-ideals, bi Γ-ideals, interior Γ-ideals) over S such that* $(F, A) \in \Omega_T(S)$ *if and only if* $\mu^+_{F(c)}(e) = 1$ *and* $\mu^-_{F(c)}(e) = -1$ *then* $(\Omega_T(S), \circ_\Gamma, \cap_R)$ *forms a complete lattice under* $\tilde{\subseteq}$.

PROOF:

For any (F, A), $(G, B) \in \Delta_T(S)$ implies that $\mu^+_{F(c)}(e) = \mu^+_{G(c)}(e) = 1$ and $\mu^-_{F(c)}(e) = \mu^-_{G(c)}(e) = -1$. As proved above $(F, A) \cap_R (G, B)$ and $(F, A) \circ_\Gamma (G, B)$ are bipolar fuzzy soft left Γ-ideals over S. Obviously, $\mu^+_{(F\cap G)(c)}(e) = 1$, $\mu^-_{(F\cap G)(c)}(e) = -1$ and $\mu^+_{(F\Gamma G)(c)}(e) = 1$, $\mu^-_{(F\Gamma G)(c)}(e) = -1$, $(F, A) \cap_R (G, B) \in \Delta_T(S)$ and $(F, A) \circ_\Gamma (G, B) \in \Delta_T(S)$. Clearly, $(F, A) \cap_R (G, B)$ is the greatest lower bound of $\{(F, A), (G, B)\}$. Now, let $c \in A \cup B$ and $x \in S$ then, we have following cases,

(i) $c \in A - B$ implies that $\mu^+_{(F\ \Gamma G)(c)}(x) = \mu^+_{(F)(c)}(x)$ and $\mu^-_{(F\ \Gamma G)(c)}(x) = \mu^-_{(F)(c)}(x)$.

(ii) $c \in B - A$ implies that $\mu^+_{(F\ \Gamma G)(c)}(x) = \mu^+_{(G)(c)}(x)$ and $\mu^-_{(F\Gamma G)(c)}(x) = \mu^-_{(G)(c)}(x)$.

(iii) $c \in A \cap B$ implies that

$$\mu^+_{(F\ \Gamma G)(c)}(x) = \sup_{x=x\gamma e} \min\{\mu^+_{F(c)}(x), \mu^+_{G(c)}(e)\}$$
$$\geq \min\{\mu^+_{F(c)}(x), \mu^+_{G(c)}(e)\}$$
$$= \mu^+_{F(c)}(x), \text{ since } \mu^+_{G(z)}(e) = 1,$$

and

$$\mu^-_{(F\ \Gamma G)(c)}(x) = \inf_{x=x\gamma e} \max\{\mu^-_{F(c)}(x), \mu^-_{G(c)}(e)\}$$
$$\leq \max\{\mu^-_{F(c)}(x), \mu^-_{G(c)}(e)\}$$
$$= \mu^-_{F(c)}(x), \text{ since } \mu^-_{G(z)}(e) = -1.$$

From above we can write,

$$\mu^+_{(F\,\Gamma G)(c)}(x) \geq \mu^+_{F(c)}(x) \text{ and}$$

$$\mu^-_{(F\,\Gamma G)(c)}(x) \leq \mu^-_{F(c)}(x), \text{ for all } x \in S.$$

This shows that $(F, A) \tilde{\subseteq} (F, A) \circ_\Gamma (G, B)$. Similarly, $(G, B) \tilde{\subseteq} (F, A) \circ_\Gamma (G, B)$. This implies that $(F, A) \circ_\Gamma (G, B)$ is an upper bound of $\{(F, A), (G, B)\}$. To show that $(F, A) \circ_\Gamma (G, B)$ is least in the sense of upper bound. Let $(P, Q) \in \Omega_T(S)$ be another upper bound of $\{(F, A), (G, B)\}$. Then $(F, A) \subseteq (P, Q)$ and $(G, B) \subseteq (P, Q)$ implies that $(F, A) \circ_\Gamma (G, B) \subseteq (P, Q) \circ_\Gamma (P, Q) \subseteq (P, Q)$. Which shows that $(F, A) \circ_\Gamma (G, B)$ is the least upper bound of $\{(F, A), (G, B)\}$. Hence we conclude that $(\Omega_T(S), \circ_\Gamma, \cap_R)$ is a complete lattice.

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