

The Orbit Graph of Finite p -Groups and Groups of Order pq

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Abstract

In this paper G denotes a finite p -group and Γ denotes a simple undirected graph. The orbit graph is a graph whose vertices are non-central orbit under group action of G on a set. Two vertices v_1 and v_2 are adjacent in the graph if $v_1g = v_2$ where $v_1, v_2 \in \Omega$, and $g \in G$. In this paper, the orbit graph of some finite p -groups and group of order pq , where p and q are relatively prime, is found. The orbit graph is determined for the group in the case that a group acts regularly on itself, acts on itself by conjugation, and acts on a set. Besides, some graph properties are found.

Keywords: Orbit Graph; p -Groups and Group of Order pq

1. Introduction

The concept of graph theory was firstly introduced in 1736 by Leonard Euler who considered Konigsberg bridge problem. Euler used a graph with vertices and edges to solve this problem. Years later, the usefulness of graph theory has been proven to a large number of diverse fields.

Next, some basic concepts of graph theory are stated.

DEFINITION 1.1¹

A graph Γ is a mathematical structure consisting of two sets namely vertices $V(\Gamma)$ and edges $E(\Gamma)$

The vertices are adjacent in Γ if each two vertices are connected by an edge. A subgraph is a graph whose vertices are subset of the vertices and edges of Γ , denoted by Γ_{sub} .

The complete graph denoted by K_n is a graph whose vertices are adjacent to each other. However, a regular graph is a graph whose vertices have the same sizes.

The following proposition is used to find the line graph in case that the graph is regular.

PROPOSITION 1.1¹

If Γ is regular with valency n , then $L(\Gamma)$ is regular with valency $2n - 2$.

The followings are some basic concepts related to graph properties that are needed in this article:

The independent set^{1,2} is a non-empty set of $V(\Gamma)$ in which there is no adjacent between two elements of a set in Γ . Thus, the independent number denoted by $\alpha(\Gamma)$ is the maximum independent in Γ . The minimum number of coloring vertices in Γ is called the chromatic number and denoted by $\chi(\Gamma)$. The maximum distance between any pair of vertices is known as diameter of Γ and denoted by $d(\Gamma)$. In addition, The clique is the maximum number of complete subgraph, denoted by $\omega(\Gamma)$ ^{1,2,3}.

In this paper, the orbit graph of finite p -groups and groups of order pq is found. First, some definitions needed are stated in the following.

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DEFINITION 1.2⁴

A p -group, where p is a prime number, is a group whose elements has an order that is a power of p .

DEFINITION 1.3⁵

Suppose G acts on a set S . The action is said to be transitive if there is only one orbit.

DEFINITION 1.4⁶

A group acts regularly on a set if the action is transitive and $stab G(x) = 1$ for all x belong to the set.

2. Preliminaries

This section provides some works that are related to graph theory.

The idea of non-commuting graphs comes from an old question of Erdős on the size of the cliques and answered in affirmative by Neumann⁷. The definition of a non-commuting graph is stated in the following.

DEFINITION 2.1⁷

A non-commuting graph is a graph whose vertices are non central elements of a group. Two vertices v_1 and v_2 are linked by an edge whenever $v_1v_2 \neq v_2v_1$.

In 2008, Abdollahi *et al.*⁸ emphasized the existence of finite bound on the sizes of complete subgraph in graph. They showed how graph theoretical properties affect the theoretical properties of group.

The conjugacy class graph was initaly introduced by Bertram⁹ in 1990. The vertices of this graph are non-central conjugacy classes in which pair of vertices are linked to each other if their sizes are not coprime. As a consequence, numerous works have been done on this graph stated as follows.

In 1992, Bianchi *et al.*¹⁰ studied the onjugacy class graph by considering the length of non- central conjugacy classes of a group are consecutive numbers.

In 1993, Chillag *et al.*¹¹ conjugacy classes are of finite size. In addition, You *et al.*¹² classified the finite groups satisfying the following property P_4 : the orders of representatives are relatively prime for any four distinct non-central conjugacy classes.

The regularity of conjugacy class graph was recently, studied by Bianchi *et al.*¹³.

In 2013, Erfanian and Tolu¹⁴ introduced conjugate graph whose vertices are non-central elements of a group in which two vertices are adjacent if they are conjugate.

3. Main Results

This section introduces our main results. First, a brief description of our work is stated in the following context.

The orbit graph of p -groups will be found in case a group acts on itself by conjugation and regularly. The orbit graph will also be determined when a group acts on a set. Our next step is to define an orbit graph, denoted by Γ_G^Ω , whose vertices are $V(\Gamma_G^\Omega) = |\Omega| - A$, where $A = \{v \in \Omega \mid vg = gv, g \in G\}$. Two elements v_1 and v_2 are adjacent in Γ_G^Ω if there exists $g \in G$ such that $v_1g = v_2$. Thus, the orbit of $v \in \Omega$ is $O(v) = \{vg \mid g \in G\}$, hence $\Omega = \cup_{v \in \Omega} O(v)$. Since Γ_G^Ω presents the orbit graph whose vertices are non-central orbits under groups action, therefore Γ_G^Ω consists of connected components of $O(v)$. It follows that

$$\Gamma_G^{\Omega_c} = \begin{cases} (i) \chi(\Gamma_G^{\Omega_c}) = \min \{cl(v_i), v_i \in \Omega\}, \\ (ii) \omega(\Gamma_G^{\Omega_c}) = \max \{cl(v_i), v_i \in \Omega\}, \\ (iii) a(\Gamma_G^{\Omega_c}) = K(\Omega) - |A|, \\ (iv) \gamma(\Gamma_G^{\Omega_c}) = K(\Omega) - |A|, \\ (v) d(\Gamma_G^{\Omega_c}) = \max \{d(v_i, v_j), v_i, v_j \in \Omega\}. \end{cases}$$

PROPOSITION 3.1

If Γ_G^Ω is a complete graph. Then $\chi(\Gamma_G^\Omega) \geq \omega(\Gamma_G^\Omega)$.

PROOF

If Γ_G^Ω is a complete graph, thus the maximum number of complete subgraphs $\omega(\Gamma_G^\Omega)$ is less than or equal to the minimum number of coloring vertices in Γ_G^Ω . Thus the inequality holds. ■

In the following, the orbit graph for finite p -groups in case of conjugate action, regular, and a group acts on a set, is found.

THEOREM 3.1

Let G be a finite p -group and acts on itself by conjugation. Then, $\Gamma_G^\Omega = \cup_{x \in G} cl(x)$.

PROOF

Suppose that $|G| = p^n$ and Γ_G^Ω is the orbit graph whose set of vertices is $V(\Gamma_G^\Omega) = |G| - |Z(G)|$. Two vertices x_1 and x_2 are adjacent in Γ_G^Ω if and only if $\{gx_1g^{-1}, g \in G\}$ where $x_1, x_2 \in G$. The group G acts on itself by conjugation then, there exists a function $\phi : G \times G \rightarrow G$ such that $\phi_g(x) = gxg^{-1}$. Hence, the conjugacy class of $x \in G$ is $\{gxg^{-1}, g \in G\}$. As known, $G = \cup_{x \in G} \text{cl}(x)$, where $\text{cl}(x)$ is the orbit of x in G . Since $\frac{G}{|\text{cl}(x)|}$, therefore $\text{cl}(x) = p^i, 1 \leq i \leq n$. As a consequence, we have complete components of $K_{\text{cl}(x)}$ for $x \in G$, the result follows. ■

PROPOSITION 3.2

If $\Gamma_{\text{sub}}^\Omega$ is a subgraph of Γ_G^Ω , then $\chi(\Gamma_G^\Omega) \geq \chi(\Gamma_{\text{sub}}^\Omega)$.

PROOF

The proof is clear since $\Gamma_{\text{sub}}^\Omega$ is a subgraph of Γ_G^Ω , thus its colored vertex are less than or equal to the number of colored vertex in Γ_G^Ω . The inequality thus holds. ■

REMARK 3.1

The graph Γ_G^Ω is a perfect graph, since $\chi(\Gamma_G^\Omega) = \omega(\Gamma_G^\Omega)$.

PROPOSITION 3.3

Let G be a finite non-abelian group of order p and let H be a normal subgroup of a group G . If G acts on H by conjugation, then $\Gamma_{G_H}^\Omega$ is a null graph.

PROOF

Suppose that $|G| = p$ and $H \triangleleft G$. If G acts on H by conjugation, then there exists $\phi : H \times G \rightarrow H$ such that $\phi_g(h) = ghg^{-1}$. The vertices of the orbit graph are $V(\Gamma_{G_H}^\Omega) = |H| - |A|$, where two distinct vertices are linked by an edge if $v_1g = v_2$, where $v_1, v_2 \in H$. Since H is a normal subgroup and two vertices adjacent if $v_1g = v_2$, thence the graph is equal to H . Therefore, the graph contains only elements of H , thus the graph is an empty graph. ■

Next, the orbit graph in the case that the group acts on the set Ω , is found.

PROPOSITION 3.4

Let G be a group acts regularly on Ω and $\frac{|G|}{|\Omega|}$. Then, $\Gamma_G^\Omega = K_p$.

PROOF

If $|G| = p$ and $\frac{|G|}{|\Omega|}$, then $|\Omega| = 1$ or p . The number vertices of this graph is $V(\Gamma_G^\Omega) = |\Omega| - |A|$. Since the action is regular, thus two vertices v_1 and v_2 of this graph are adjacent if there exists $g \in G$ such that $v_1g = v_2$. By Orbit Stabilizer Theorem, the size of conjugacy class under the action divides the order of $|G|$, thus Γ_G^Ω consists of complete graph of K_p . ■

Next, we find Γ_G^Ω in the case that the group acts on the set Ω transitively.

PROPOSITION 3.5

Let G be a group of order p . If G acts transitively on Ω , then $\Gamma_G^\Omega = K_p$.

PROOF

If $|G| = p$ in which G acts transitively on Ω , then there exists only one orbit, namely. Hence the graph is complete (i.e $\Gamma_G^\Omega = K_p$). ■

COROLLARY 3.1

Let G be a finite group of order p , G acts transitively on Ω . Then the graph Γ_G^Ω is a connected graph.

PROOF

The proof is clear since the graph is a complete graph K_p , thus all vertices of the graph are adjacent, hence the graph is connected. ■

Next, we find the orbit graph of group of order pq .

PROPOSITION 3.6

Let G be a group of order pq , where p and q are relatively prime and let G act on itself by conjugation. Then

$$\Gamma_G^\Omega = \bigcup_{\substack{i=1 \\ x \in G}} K_{\text{cl}(x_i)}$$

PROOF

Suppose that $|G| = pq$ and G acts on itself by conjugation, Therefore, the conjugacy class of x in G is $\text{cl}(x) = \{x, g \in G : gxg^{-1}\}$. Thus, the number of vertices $V(\Gamma_G^\Omega) = |G|$. Since $G = \bigcup_{x \in G} \text{cl}(x)$ and $\frac{|G|}{|\text{cl}(x)|}$, hence $|\text{cl}(x)| = p, q$ or pq . Two vertices are adjacent if they are conjugate, thus Γ_G^Ω consists of complete components of sizes of conjugacy classes. Henceforth, $\Gamma_G^\Omega = \bigcup_{\substack{i=1 \\ x \in G}} K_{\text{cl}(x_i)}$. ■

In the following, the orbit graph of group of order pq in the case the group acts on Ω .

THEOREM 3.4

Let G be a finite group of order pq , where p and q are distinct primes. Let Ω be the set of all subsets of commuting elements in the form (a, b) of size two in G . If G acts on Ω by conjugation, then, $\Gamma_G^\Omega = K_{|\Omega|}$.

PROOF

Assume that G is a finite group of order pq and Ω be the set of all commuting elements of the form of (a, b) of size two. Thus $V(\Gamma_G^\Omega) = |\Omega| - |A|$, where $A = \{x \in \Omega : xg = gx, g \in G\}$. If G acts on Ω by conjugation, then $\text{cl}(x) = \{gxg^{-1}, g \in G\}$. Since the size of orbit under the action divides the order of G , sing adjacency of vertices, Γ_G^Ω consists of complete components with number of adjacent verices p, q or pq . The proof then follows. ■

REMARK 3.2

The graph in Theorem 3.4 is a connected gra.

PROPOSITION 3.7

Let G be a finite group of order pq and be the set of all subsets of commuting elements in the form (a, b) of size two. Then, The chromatic number $\chi(\Gamma_G^\Omega)$ and the clique number $\omega(\Gamma_G^\Omega)$ are equal to $|\Omega|$.

PROOF

It is obvious since we have a complete graph $K_{|\Omega|}$. The proof then follows. ■

COROLLARY 3.2

The graph Γ_G^Ω is a perfect graph.

PROOF

The proof follows from Proposition 3.7. ■

4. Conclusion

In this paper, the orbit graph for finite p -groups and group of order pq , is found under some groups action including conjugate regular. In addition, some graph properties including the chromatic number, independent number, clique number and dominating number are determined.

5. Acknowledgement

The first author would like to acknowledge Ministry of Hiher Education in Libya.

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