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Approximate symmetric solution of dual fuzzy systems regarding two different fuzzy multiplications

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Abstract

We consider two types of dual fuzzy systems with respect to two different fuzzy multiplications and propose an approach for computing an approximate nonnegative symmetric solution of some dual fuzzy linear system of equations. We convert the $m \times n$ dual fuzzy linear system to two $m \times n$ real linear systems by considering equality of the median intervals of the left and right sides of the dual fuzzy system. Then, the real systems are solved, when the solutions does not satisfy nonnegative fuzziness conditions, an appropriate constrained least squares problem is solved. We finally present some computational algorithms and illustrate their effectiveness by solving some randomly generated consistent as well as inconsistent systems.

Keywords: *LR* fuzzy numbers, Triangular fuzzy numbers, Dual fuzzy systems, Median interval defuzzification.

Introduction

Systems of linear equations arise from various areas of science and engineering. Since many real world systems are too complex to be defined in precise terms, imprecision is admittedly present. Analyzing such systems demands the use of fuzzy analysis. Therefore, the fuzzy concept proposed by Zadeh (Chang & Zadeh, 1972; Zadeh, 1972; Zadeh, 1975) is deemed to be quite useful in such applications. Thus, the need for solving linear systems whose parameters are all or partially represented by fuzzy numbers persists.

A general model for solving an arbitrary square fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy vector was first proposed by Friedman *et al*. (1998). For computing a solution, they used the embedding method and replaced the original fuzzy $n \times n$ linear system by a $2n \times 2n$ crisp linear system. The approach in (Friedman *et al*., 1998), however, does not fully address solutions of general fuzzy systems.

Other numerical procedures for solving fuzzy linear systems such as Jacobi, Gauss-Seidel, Adomiam decomposition method and SOR iterative method can be seen in (Allahviranloo *et al*., 2006).In (Dehghan & Hashemi, 2007), Dehghan *et al*. also proposed some iterative techniques. Abbasbandy *et al*. (2005) solved $m \times n\,(m \le n)$ original fuzzy linear system using a $2m \times 2n$ real system, and Allahviranloo (2008) studied finding least squares solution of an overdetermined $(m > n)$ fuzzy linear system.

In the more general system of fully fuzzy linear system of equations (FFLSE), both the coefficient matrix and the right hand sides are considered to be fuzzy and a fuzzy solution is desired. A conceivable method for solving an FFLSE is to consider real systems corresponding to α -cuts whose coefficient matrices and right hand side vectors, the A and b are extracted from the fuzzy coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$ and fuzzy right hand side vector $\tilde{b} = (\tilde{b}_i)$, respectively. Buckley and Qu (1991) gave a method to compute the set of solutions. For A nonsingular, $x = A^{-1}b$ with a positive membership, is a solution of the corresponding system and

$$\{x \mid x = A^{-1}b, A = (a_{ij}), b = (b_i),$$
$$a_{ij} \in supp(\tilde{a}_{ij}), b_i \in supp(\tilde{b}_i), \text{ for all } i, j\},$$

is the set of solutions where $supp\,(.)$ denotes the support of fuzzy number. Based on this fact, they discussed the theoretical aspects of this class of problems, extended several methods for this class and proved their equivalence. But their approach is not practical, because a large number of real systems may be required to be considered for the fully fuzzy linear system using α -cuts.

In recent years, Dehghan *et al*. (2006c) also considered solving such systems. They applied some well-known iterative methods such as Jacobi, Gauss-Seidel, SOR, AOR on the FFLSE. Moreover, they focused on the use of Adomian decomposition method for solving the FFLSE (Dehghan & Hashemi, 2006a). They also used classical methods like Cramer's rule, Guassian elimination and LU decomposition to find the approximate solution (Dehghan *et al*., 2006b). To guarantee that the produced solution would be fuzzy, at times they applied linear programming models. In (Ezzati *et al*.(2012), an approximate solution of FFLSE in which all the components of the coefficient matrix are either nonnegative or nonpositive, were computed by solving an appropriate least squares problems.

Muzzioli *et al*. (2006) have extended Buckley and Qu's method to a more general fuzzy system of equations $A_1x + b_1 = A_2x + b_2$, with A_1 , A_2 square matrices of coefficients and b_1, b_2 fuzzy number vectors. They gave some conditions under which a vector solution of the system exists. Moreover, they showed that the linear systems $Ax = b$ and $A_1x + b_1 = A_2x + b_2$, with

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$A = A_1 - A_2$ and $b = b_2 - b_1$ have the same vector solutions and they introduced an computational algorithm to find the vector solution. Some necessary and sufficient conditions for the solution existence of the dual fuzzy linear system of equations were given in (Ming *et al*., 2000). Some other results can be found in (Abbasandy *et al*., 2006; Ezzati & Abbasbandy, 2007; Ezzati, 2008).

Here, we intend to compute an approximate symmetric solution of the dual fuzzy system of equations using median interval defuzzification, an approach of Bodjanova in (Bodjanova, 2005). By considering equality of the median intervals of the left and right sides of the system, we convert the dual fuzzy linear system to two real systems of equations, one being concerned with the cores and the other being related to the spreads. Then, these real systems are solved. There are several approaches in the literature for solving real linear systems of equations. These systems can be solved by using the inverse or pseudinverse of the coefficient matrices (Friedman *et al*., 1998;Dehghan *et al*., 2006c) or the different decompositions (Dehghan & Hashemi, 2006a). If each of them is inconsistent or the solution of the spread system does not satisfy the fuzziness conditions, we find an approximate solution by solving a respective least squares problem.

In the preliminaries section, we present basic definitions and the notation of median interval for a solution. The next two sections discuss the approximate solution of the dual fuzzy linear system of equations with respect to fuzzy multiplication defined by Kauffman and Gupta (1991) and Dubois and Prade (1980). Some numerical results are provided in the section of numerical examples. Finally, our concluding remarks are given.

Preliminaries

In this section, we review some concepts and results of fuzzy numbers been in the literature (Kauffman & Gupta, 1991; Zimmermann, 1991).

Definition 1: A fuzzy set $\tilde{A}$ with the membership function $\mu_{\tilde{A}}(x): \Re \to [0,1]$ is a fuzzy number if the following properties are satisfied:

- $\mu_{\tilde{A}}$ is an upper semi-continuous function on $\Re$.
- There are real numbers *a, b, c* and *d* such that $a \le b \le c \le d$ and
  - $\mu_{\tilde{A}}(x)$ is a monotonically increasing function on $[a,b]$,
  - $\mu_{\tilde{A}}(x)$ is a monotonically decreasing function on $[c,d]$,
  - $\mu_{\tilde{A}}(x) = 1$, for all x in $[b,c]$.
- $\mu_{\tilde{A}}(x) = 0$, for x outside of the interval $[a,d] \subset \Re$.

We denote the set of all fuzzy numbers by F.

In addition, a fuzzy number $\tilde{A}$ is called positive (negative), shown as $\tilde{A} > 0$ $(\tilde{A} < 0)$, if its membership functin
$\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0$, for all $x < 0$ ($x > 0$).

Definition 2 : A fuzzy number $\tilde{A}$ is said to be an *LR* fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), & x \le a,\ \alpha > 0, \\ R(\frac{x-a}{\beta}), & x \ge a,\ \beta > 0, \end{cases} \quad (1)$$

where a is the core of $\tilde{A}$, α and β are left and right spreads, respectively, and the function $L(.)$, which is called left shape function, satisfies:

- $L(x) = L(-x)$,
- $L(0) = 1$ and $L(1) = 0$,
- $L(x)$ is nonincreasing on $[0,\infty)$.

The definition of right shape function $R(.)$ is similar to that of $L(.)$.

The core, left and right spreads, and the shape function of an LR fuzzy number $\tilde{A}$ are symbolically shown as $\tilde{A} = (a,\alpha,\beta)_{LR}$. Clearly, $\tilde{A} = (a,\alpha,\beta)_{LR}$ is positive if and only if $a - \alpha > 0$ (since $L(1) = 0$), and $\tilde{A} = (a,\alpha,\beta)_{LR}$ is a symmetric fuzzy number if and only if $\alpha = \beta$.

We assume that a fuzzy number $\tilde{A} = \langle a,b,c,d \rangle$ can be expressed in the form,

$$\tilde{A}(x) = \begin{cases} g(x), & x \in [a,b), \\ 1, & x \in [b,c], \\ h(x), & x \in (c,d], \\ 0, & otherwise, \end{cases} \quad (2)$$

where *a, b, c* and *d* are real numbers such that $a < b \le c < d$, g is an increasing real valued function and right continuous and h is a decreasing real valued function continuous and left continuous. A fuzzy number $\tilde{A}$ with the shape function g and h defined by

$$g(x) = \left(\frac{x-a}{b-a} \right)^n, \quad (3)$$

and

$$h(x) = \left(\frac{d-x}{d-c} \right)^n, \quad (4)$$

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respectively, where $n > 0$, is denoted by $\tilde{A} = (a,b,c,d)_n$.

Definition 3: A fuzzy number $\tilde{A} = (a,b,c,d)$ is known as a trapezoidal fuzzy number, if $n = 1$ in (3) and (4). Also, when $n = 1$ and $b = c$, $\tilde{A}$ is known as a triangular fuzzy number to which we always refer here by $\tilde{A} = (a,b,c)$.

Each fuzzy number $\tilde{A}$ described by (1) has the following α -cuts, $\tilde{A}_\alpha = [a_\alpha, b_\alpha]$, $a_\alpha, b_\alpha \in \Re$, $\alpha \in [0,1]$:

- $\tilde{A}_\alpha = [g^{-1}(\alpha), h^{-1}(\alpha)],\ for\ all\ \alpha \in (0,1),$
- $\tilde{A}_0 = [a,d],$
- $\tilde{A}_1 = [b,c].$

If $\tilde{A} = \langle a,b,c,d \rangle_n$, then for all $\alpha \in [0,1]$,

$$\tilde{A}_\alpha = [a + \alpha^{\frac{1}{n}}(b-a), d - \alpha^{\frac{1}{n}}(d-c)]. \quad (5)$$

Note that the Hausdorff distance is used to compute the distance between two fuzzy numbers.

Definition 4:Let A and B be fuzzy numbers. The Hausdorff distance between A and B is defined to be:

$$d_H(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right). \quad (6)$$

Now, we review the definition of the median interval and some corresponding results (Bodjanova, 2005).

Definition 5:Let $\tilde{A}$ be a fuzzy number as gives by (1). Let $m_L \in (a,b)$ and $m_R \in (c,d)$ be such that

$\int_a^{m_L} \tilde{A}(x)dx = \int_{m_L}^{b} \tilde{A}(x)dx$ and

$$\int_c^{m_R} \tilde{A}(x)dx = \int_{m_R}^{d} \tilde{A}(x)dx.$$

Then, $M_e(\tilde{A}) = [m_L, m_R]$ is called the median interval (interval-valued median) of $\tilde{A}$.

Proposition 1:Let $\tilde{A} = \langle a,b,c,d \rangle_n$. Then, $M_e(\tilde{A}) = [m_L, m_R]$, where,

$$m_L = a + \frac{(b-a)}{\sqrt[n+1]{2}} \quad (7)$$

and

$$m_R = d - \frac{(d-c)}{\sqrt[n+1]{2}}. \quad (8)$$

Corollary 1:Let $\tilde{A} = \langle a,b,c,d \rangle_n$. Then, $M_e(\tilde{A}) = A_\alpha$ and $\alpha \in (0.5,1)$, where $\alpha = 2^{-\frac{n}{n+1}}$.

Corollary 2:Let $\tilde{A} = \langle a,b,c,d \rangle$ be a trapezoidal fuzzy number. Then, $M_e(\tilde{A}) = \tilde{A}_\alpha$, where $\alpha = 0.707$.

In general, If $\tilde{A}$ can not be expressed in the form $\langle a,b,c,d \rangle_n$, then $M_e(\tilde{A})$ is not an α -cut of $\tilde{A}$; see (Bodjanova, 2005).

The dual fuzzy linear system of equations is defined as follows.

Definition 6:The system

$$(\tilde{A} \otimes \tilde{x}) \oplus \tilde{b} = (\tilde{C} \otimes \tilde{x}) \oplus \tilde{d}, \quad (9)$$

where the components of $\tilde{A} = (a_{ij})_{m\times n}$ and $\tilde{C} = (c_{ij})_{m\times n}$, the coefficient matrices, and $\tilde{b} = (b_i)_{m\times 1}$ and $\tilde{d} = (d_i)_{m\times 1}$, the constant vectors, are fuzzy numbers is called Dual Fuzzy Linear System of Equations (DFLSE).

Remark 1:There is no inverse element for an arbitrary fuzzy number; it means

$\tilde{x} \oplus (-\tilde{x}) \neq 0.$

So, the system (9) is not equivalent to

$(\tilde{A} - \tilde{C}) \otimes \tilde{x} = \tilde{d} - \tilde{b}.$

DFLSE and its approximate symmetric solution with respect to the fuzzy multiplication defined in (Kauffman & Gupta, 1991):

In (Kauffman & Gupta, 1991), Kauffman and Gupta designed the following exact formulas for the addition and subtraction of the triangular fuzzy numbers and an approximate formula for the multiplication. Let $\tilde{A} = (a,b,c)$ and $\tilde{B} = (x,y,z)$ be two triangular fuzzy numbers then

- Addition:

$$\tilde{A} \oplus \tilde{B} = (a,b,c) \oplus (x,y,z) = (a+x, b+y, c+z). \quad (10)$$

- Subtraction:

$$\tilde{A} \Theta \tilde{B} = (a,b,c) \Theta (x,y,z) = (a-z, b-y, c-x). \quad (11)$$

- Multiplication:

Let $\tilde{B}$ be nonnegative then

$$\tilde{A} \otimes_K \tilde{B} = (a,b,c) \otimes_K (x,y,z) \cong \begin{cases} (ax,by,cz), & a \geq 0, \\ (az,by,cz), & a < 0, c > 0, \\ (az,by,cx), & c \leq 0. \end{cases} \quad (12)$$

Consider the dual fuzzy linear system (9) with the multiplication (12) or equivalently, the $m \times n$ linear system of equations,

$$\begin{cases} (\tilde{a}_{11} \otimes_K \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes_K \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes_K \tilde{x}_n) \oplus \tilde{b}_1 = \\ \quad (\tilde{c}_{11} \otimes_K \tilde{x}_1) \oplus (\tilde{c}_{12} \otimes_K \tilde{x}_2) \oplus \cdots \oplus (\tilde{c}_{1n} \otimes_K \tilde{x}_n) \oplus \tilde{d}_1, \\ \quad \vdots \\ (\tilde{a}_{m1} \otimes_K \tilde{x}_1) \oplus (\tilde{a}_{m2} \otimes_K \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{mn} \otimes_K \tilde{x}_n) \oplus \tilde{b}_m = \\ \quad (\tilde{c}_{m1} \otimes_K \tilde{x}_1) \oplus (\tilde{c}_{m2} \otimes_K \tilde{x}_2) \oplus \cdots \oplus (\tilde{c}_{mn} \otimes_K \tilde{x}_n) \oplus \tilde{d}_m, \end{cases}$$

(13)

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where $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$, $1 \le i \le m$, $1 \le j \le n$, the coefficient matrices components, and $\tilde{b}_i$ and $\tilde{d}_i$, $1 \le i \le m$, components of the constant vectors are fuzzy numbers.We give the following definition characterizing an approximate solution of DFLSE (9) by median interval defuzzification.

Definition 7: A fuzzy vector $\tilde{x}$ is called an approximate solution of DFLSE if the median intervals of the left and right sides of DFLSE are equal, that is,

$$M_e(((\tilde{A} \otimes_K \tilde{x}) \oplus \tilde{b})_i) = M_e(((\tilde{C} \otimes_K \tilde{x}) \oplus \tilde{d})_i), \quad i = 1, \ldots, m. \quad (14)$$

Here, we assume that all parameters in (13) are triangular fuzzy numbers defined by Definition 3, that is,

$$\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}), \quad \tilde{c}_{ij} = (e_{ij}, f_{ij}, g_{ij}), \quad 1 \le i \le m, 1 \le j \le n,$$
$$\tilde{b}_i = (b_{1i}, b_{2i}, b_{3i}), \quad \tilde{d}_i = (d_{1i}, d_{2i}, d_{3i}), \quad 1 \le i \le m,$$
(15)

where a_{ij}, e_{ij}, b_{1i} and d_{1i}, and c_{ij}, g_{ij}, b_{3i} and d_{3i} denote the beginning and ending points of the 0-cuts of $\tilde{a}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_i$ and $\tilde{d}_i$, respectively.

Notation 1: Set

$$A = (a_{ij})_{m \times n}, \; B = (b_{ij})_{m \times n}, \; C = (c_{ij})_{m \times n},$$
$$E = (e_{ij})_{m \times n}, \; F = (f_{ij})_{m \times n}, \; G = (g_{ij})_{m \times n},$$
$$b_1 = (b_{1i})_{m \times 1}, \; b_2 = (b_{2i})_{m \times 1}, \; b_3 = (b_{3i})_{m \times 1},$$
$$d_1 = (d_{1i})_{m \times 1}, \; d_2 = (d_{2i})_{m \times 1}, \; d_3 = (d_{3i})_{m \times 1}.$$

Notation 2: We break up the matrix A into two $m \times n$ matrices such that their addition is A. Let $A^+ = (a_{ij}^+)_{m \times n}$ and $A^- = (a_{ij}^-)_{m \times n}$, where,

$$a_{ij}^+ = \begin{cases} a_{ij} & a_{ij} \ge 0 \\ 0 & a_{ij} < 0 \end{cases}, \quad a_{ij}^- = \begin{cases} 0 & a_{ij} \ge 0 \\ a_{ij} & a_{ij} < 0 \end{cases}.$$

Then $A^+ + A^- = A$. We also break up the matrices C, E and G into two $m \times n$ matrices, similarly. Let $C^+ = (c_{ij}^+)$, $C^- = (c_{ij}^-)$, $E^+ = (e_{ij}^+)$, $E^- = (e_{ij}^-)$, $G^+ = (g_{ij}^+)$ and $G^- = (g_{ij}^-)$ be $m \times n$ matrices, where,

$$c_{ij}^+ = \begin{cases} c_{ij} & c_{ij} \ge 0 \\ 0 & c_{ij} < 0 \end{cases}, \quad c_{ij}^- = \begin{cases} 0 & c_{ij} \ge 0 \\ c_{ij} & c_{ij} < 0 \end{cases},$$

$$e_{ij}^+ = \begin{cases} e_{ij} & e_{ij} \ge 0 \\ 0 & e_{ij} < 0 \end{cases}, \quad e_{ij}^- = \begin{cases} 0 & e_{ij} \ge 0 \\ e_{ij} & e_{ij} < 0 \end{cases},$$

$$g_{ij}^+ = \begin{cases} g_{ij} & g_{ij} \ge 0 \\ 0 & g_{ij} < 0 \end{cases}, \quad g_{ij}^- = \begin{cases} 0 & g_{ij} \ge 0 \\ g_{ij} & g_{ij} < 0 \end{cases}.$$

Proposition 3: The system (9) with the multiplication (12) is equivalent to

$$(A^+x + A^-z + b_1, By + b_2, C^+z + C^-x + b_3)$$
$$= (E^+x + E^-z + d_1, Fy + d_2, G^+z + G^-x + d_3), \quad (16)$$

where $\tilde{x}_j = (x_j, y_j, z_j)$, $j = 1, \ldots, n$, are nonnegative and $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$, $z = (z_1, \ldots, z_n)^T$.

Proof. From (12),

$$(\tilde{A} \otimes_K \tilde{x})_i = \sum_{j=1}^{n} (\tilde{a}_{ij} \otimes_K \tilde{x}_j)$$
$$= \sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}) \otimes_K (x_j, y_j, z_j)$$
$$= \sum_{j, a_{ij} \ge 0} (a_{ij}x_j, b_{ij}y_j, c_{ij}z_j)$$
$$+ \sum_{j, a_{ij} \le 0, c_{ij} > 0} (a_{ij}z_j, b_{ij}y_j, c_{ij}z_j)$$
$$+ \sum_{j, c_{ij} \le 0} (a_{ij}z_j, b_{ij}y_j, c_{ij}x_j)$$
$$= \left(\sum_{j, a_{ij} \ge 0} a_{ij}x_j + \sum_{j, a_{ij} < 0} a_{ij}z_j, \sum_{j=1}^{n} b_{ij}y_j, \sum_{j, c_{ij} \ge 0} c_{ij}z_j + \sum_{j, c_{ij} < 0} c_{ij}x_j \right),$$
$$i = 1, \ldots, m.$$

Similarly,

$$(\tilde{C} \otimes_K \tilde{x})_i = \sum_{j=1}^{n} (\tilde{c}_{ij} \otimes_K \tilde{x}_j)$$
$$= \left(\sum_{j, e_{ij} \ge 0} e_{ij}x_j + \sum_{j, e_{ij} < 0} e_{ij}z_j, \sum_{j=1}^{n} f_{ij}y_j, \sum_{j, g_{ij} \ge 0} g_{ij}z_j + \sum_{j, g_{ij} < 0} g_{ij}x_j \right),$$
$i = 1, \ldots, m$,

and by (10), the system (9) is equivalent to

$$(A^+x + A^-z + b_1, By + b_2, C^+z + C^-x + b_3)$$
$$= (E^+x + E^-z + d_1, Fy + d_2, G^+z + G^-x + d_3). \; \lozenge$$

We have the following result, when all parameters in (13) are triangular fuzzy numbers.

Proposition 4: A fuzzy vector $(\tilde{x}_1, \ldots, \tilde{x}_n)^T$ given by $\tilde{x}_j = (x_j, y_j, z_j)$, $j = 1, \ldots, n$, is a nonnegative approximate solution of (9) with the multiplication (12) if and only if it satisfies

$$\begin{cases} (A^+ - E^+)x + k(B - F)y + (A^- - E^-)z = k(d_2 - b_2) + d_1 - b_1, \\ (C^- - G^-)x + k(B - F)y + (C^+ - G^+)z = k(d_2 - b_2) + d_3 - b_3, \end{cases}$$

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where $x=(x_1,...,x_n)^T$, $y=(y_1,...,y_n)^T$, $z=(z_1,\ldots,z_n)^T$ and $k=1+\sqrt{2}$.

Proof. Since the systems (9) and (16) are equivalent (Proposition 3), the proof is simply obtained by equating the median intervals of left and right sides of (16). ◊

Here, to find nonnegative approximate solutions of DFLSE which are symmetric triangular fuzzy numbers, we have the following results.

Theorem 1: A fuzzy vector $(\tilde{x}_1,\ldots,\tilde{x}_n)^T$ given by $\tilde{x}_j=(x_j,y_j,z_j)$, $j=1,\ldots,n$ is a nonnegative approximate symmetric triangular solution of (9) if and only if $x=(x_1,..., x_n)^T$ and $y=(y_1,..., y_n)^T$ satisfy the system,

$$\begin{cases}(|A|-|E|)x+(k(B-F)+2(A^- - E^-))y=k(d_2-b_2)+d_1-b_1,\\(|G|-|C|)x+(k(B-F)+2(C^+ - G^+))y=k(d_2-b_2)+d_3-b_3,\end{cases} \quad (18)$$

where $k=1+\sqrt{2}$ and $|.|$ denotes the absolute value of a matrix, for instance $|A|=\left(\,|a_{ij}|\,\right)_{m\times n}$. $z=(z_1,...,z_n)^T$ is given by $z=2y-x$.

Proof. By solving the system,

$$\begin{cases}(A^+ - E^+)x+k(B-F)y+(A^- - E^-)z=k(d_2-b_2)+d_1-b_1,\\(C^- - G^-)x+k(B-F)y+(C^+ - G^+)z=k(d_2-b_2)+d_3-b_3,\\x-2y+z=0,\end{cases}$$

where $k=1+\sqrt{2}$, we can obtain (18). ◊

The crisp system (18) can be solved using any existence method for solving the real linear systems. The obtained results would form a nonnegative approximate symmetric fuzzy solution for (9) if solutions of (18), x_j and y_j, satisfy $x_j\ge 0$ and $x_j-y_j\le 0$, for $j=1,..., n$. Otherwise, we may try to find approximate solutions for (9). For this purpose, we consider the following constrained least squares problem,

$$\begin{aligned}&\min_w z_{core}=\left\|\overline{C}w-\overline{d}\right\|_2\\&s.t.\\&\quad w_{n+i}-w_i\ge 0,\quad i=1,\ldots,n,\\&\quad w_i\ge 0,\qquad i=1,\ldots,n,\end{aligned} \quad (19)$$

where $w=(x_1,\ldots,x_n,y_1,\ldots,y_n)^T$,

$$\overline{C}=\begin{pmatrix}|A|-|E| & (1+\sqrt{2})(B-F)+2(A^- - E^-)\\ \\ |G|-|C| & (1+\sqrt{2})(B-F)+2(C^+ - G^+)\end{pmatrix}_{2m\times 2n}$$

and

$$\overline{d}=\begin{pmatrix}d_1-b_1+(1+\sqrt{2})(d_2-b_2)\\ \\ d_3-b_3+(1+\sqrt{2})(d_2-b_2)\end{pmatrix}_{2m\times 1}. \quad (21)$$

Note that the problem (19) always has solutions, since a nonnegative objective functions is minimized on a convex feasible space.

Remark 2: First, we solve the real system (18). If it lacks the desirable solution, that is there exists an index j such that $x_j<0$ or $x_j>y_j$, then the least squares problem (19) is solved for which the value of the objective function is non-zero.

Ghanbari *et al*. (2010) proposed an approach to guarantee that the obtained solution be fuzzy numbers. Here, we follow the same approach and provide the corresponding necessary adjustments for the optimization problems (19).

Remark 3: In the least squares problem (19), we may consider the numbers $w_{n+j}=y_j$ or $\delta_j=w_{n+j}-w_j=y_j-x_j$, for $j=1,\ldots,n$, with small positive values, say 10^{-20}. In this case, the nonnegative symmetric triangular fuzzy numbers (x_j,y_j,z_j) are very close to zero (or the crisp numbers y_j). Thus, a positive parameter ε can be taken such that $y_j\ge\varepsilon$ or $\delta_j\ge\varepsilon$, for all j. This choice guarantees that each fuzzy number has a positive core at least as much as ε. Therefore, instead of problem (19), we consider the problem,

$$\begin{aligned}&\min_w z_{core}=\left\|\overline{C}w-\overline{d}\right\|_2\\&s.t.\\&\quad w_{n+i}-w_i\ge\varepsilon,\quad i=1,\ldots,n,\\&\quad w_i\ge 0,\qquad i=1,\ldots,n,\end{aligned} \quad (22)$$

where $\varepsilon>0$ is a user-defined parameter.

We are ready to give an algorithm for solving the DFLSE regarding the fuzzy multiplication defined in (Kauffman & Gupta, 1991).

Algorithm 1: Find an approximate symmetric triangular solution for the DFLSE with the fuzzy multiplication (12).

A. Give $\varepsilon>0$ to be used for (22).
B. Solve the system (18). **If** the system (18) has no solution (consequently the system (9) does not have a solution) **or** y or x, the solutions of (18), has at least one negative component **or** there exist an index j such that $x_j-y_j>0$ **then** solve the (20) optimization problem (22).
C. Compute $z_j=2y_j-x_j$ for all $j=1,\ldots,n$.

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D. The approximate symmetric solution of (9) is $\tilde{x}=\left(\tilde{x}_1,\ldots,\tilde{x}_n\right)^T$, where $\tilde{x}_j=(x_j,y_j,z_j)$, $j=1,\ldots,n$, $x=\left(x_1,\ldots,x_n\right)^T$, $y=\left(y_1,\ldots,y_n\right)^T$ and $z=\left(z_1,\ldots,z_n\right)^T$.

E. *Stop.*

$FFLSE$ and its approximate symmetric solution with respect to the fuzzy multiplication defined in (Dubois & Prade, 1980):

Dubois and Prade (1980) designed the following exact formulas for the addition of LR fuzzy numbers and scalar multiplication. They also introduced an approximate formula for multiplying the LR fuzzy numbers. For two LR fuzzy numbers $\tilde{A}=(a,\alpha,\beta)_{LR}$ and $\tilde{B}=(b,\gamma,\delta)_{LR}$, we have:

- Addition:

$$\tilde{A}\oplus\tilde{B}=(a,\alpha,\beta)_{LR}\oplus(b,\gamma,\delta)_{LR}=(a+b,\alpha+\gamma,\beta+\delta)_{LR}. \quad (23)$$

- Multiplication:

If $\tilde{A}>0$ and $\tilde{B}>0$, then

$$\begin{aligned}\tilde{A}\otimes_{DP}\tilde{B}&=(a,\alpha,\beta)_{LR}\otimes_{DP}(b,\gamma,\delta)_{LR}\\&\cong(ab,a\gamma+b\alpha,a\delta+b\beta)_{LR}.\end{aligned} \quad (24)$$

If $\tilde{A}<0$ and $\tilde{B}>0$, then

$$\begin{aligned}\tilde{A}\otimes_{DP}\tilde{B}&=(a,\alpha,\beta)_{LR}\otimes_{DP}(b,\gamma,\delta)_{LR}\\&\cong(ab,b\alpha-a\delta,b\beta-a\gamma)_{LR}.\end{aligned} \quad (25)$$

- Scalar multiplication:

$$\lambda\otimes\tilde{A}=\lambda\otimes(a,\alpha,\beta)_{LR}=\begin{cases}(\lambda a,\lambda\alpha,\lambda\beta)_{LR}, & \lambda>0,\\(\lambda a,-\lambda\beta,-\lambda\alpha)_{LR}, & \lambda<0.\end{cases} \quad (26)$$

Consider the dual fuzzy linear system (9) or equivalently, the $m\times n$ linear system of equations,

$$\begin{cases}(\tilde{a}_{11}\otimes_{DP}\tilde{x}_1)\oplus(\tilde{a}_{12}\otimes_{DP}\tilde{x}_2)\oplus\cdots\oplus(\tilde{a}_{1n}\otimes_{DP}\tilde{x}_n)\oplus\tilde{b}_1=\\\qquad(\tilde{c}_{11}\otimes_{DP}\tilde{x}_1)\oplus(\tilde{c}_{12}\otimes_{DP}\tilde{x}_2)\oplus\cdots\oplus(\tilde{c}_{1n}\otimes_{DP}\tilde{x}_n)\oplus\tilde{d}_1\\\qquad\vdots\\(\tilde{a}_{m1}\otimes_{DP}\tilde{x}_1)\oplus(\tilde{a}_{m2}\otimes_{DP}\tilde{x}_2)\oplus\cdots\oplus(\tilde{a}_{mn}\otimes_{DP}\tilde{x}_n)\oplus\tilde{b}_m=\\\qquad(\tilde{c}_{m1}\otimes_{DP}\tilde{x}_1)\oplus(\tilde{c}_{m2}\otimes_{DP}\tilde{x}_2)\oplus\cdots\oplus(\tilde{c}_{mn}\otimes_{DP}\tilde{x}_n)\oplus\tilde{d}_m,\end{cases} \quad (27)$$

where $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$, $1\le i\le m$, $1\le j\le n$, the coefficient matrices components, and $\tilde{b}_i$ and $\tilde{d}_i$, $1\le i\le m$, components of the constant vectors are fuzzy numbers.

We give the following definition characterizing an approximate solution of DFLSE (9) by median interval defuzzification.

Definition 8: A fuzzy vector $\tilde{x}$ is called an approximate solution of the DFLSE if the median intervals of the left and right sides of the DFLSE are equal, that is,

$$M_e(((\tilde{A}\otimes_{DP}\tilde{x})\oplus\tilde{b})_i)=M_e(((\tilde{C}\otimes_{DP}\tilde{x})\oplus\tilde{d})_i),\quad i=1,\ldots,m. \quad (28)$$

Here, we assume that all parameters in (27) are nonnegative LR fuzzy numbers, that is,

$\tilde{a}_{ij}=(a_{ij},m_{ij},n_{ij}),\quad \tilde{c}_{ij}=(c_{ij},e_{ij},f_{ij}),\quad 1\le i\le m, 1\le j\le n,$

$\tilde{b}_i=(b_i,b_{iL},b_{iR}),\quad \tilde{d}_i=(d_i,d_{iL},d_{iR}),\quad 1\le i\le m,$

where m_{ij}, e_{ij}, b_{iL} and d_{iL}, and n_{ij}, f_{ij}, b_{iR} and d_{iR} denote left spreads and right spreads of $\tilde{a}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_i$ and $\tilde{d}_i$, respectively. Let

$$A=\left(a_{ij}\right)_{m\times n},\ M=\left(m_{ij}\right)_{m\times n},\ N=\left(n_{ij}\right)_{m\times n},$$
$$C=\left(c_{ij}\right)_{m\times n},\ E=\left(e_{ij}\right)_{m\times n},\ F=\left(f_{ij}\right)_{m\times n},$$
$$b=\left(b_i\right)_{m\times 1},\ b_L=\left(b_{iL}\right)_{m\times 1},\ b_R=\left(b_{iR}\right)_{m\times 1},$$
$$d=\left(d_i\right)_{m\times 1},\ d_L=\left(d_{iL}\right)_{m\times 1},\ d_R=\left(d_{iR}\right)_{m\times 1}.$$

Proposition 5: The system (9) with the multiplication (24) is equivalent to

$$\begin{aligned}&(Ax+b,Ay+Mx+b_L,Az+Nx+b_R)\\&=(Cx+d,Cy+Ex+d_L,Cz+Fx+d_R),\end{aligned} \quad (30)$$

where $\tilde{x}_j=(x_j,\alpha_j,\beta_j)$, $j=1,\ldots,n$, and $x=(x_1,\ldots,x_n)^T$, $y=(\alpha_1,\ldots,\alpha_n)^T$ and $z=(\beta_1,\ldots,\beta_n)^T$.

Proof: From (24),

$$(\tilde{a}_{ij}\otimes\tilde{x}_j)\oplus\tilde{b}_i=(a_{ij}x_j+b_i,a_{ij}\alpha_j+m_{ij}x_j+b_{iL},a_{ij}\beta_j+n_{ij}x_j+b_{iR}),$$

$$(\tilde{c}_{ij}\otimes\tilde{x}_j)\oplus\tilde{d}_i=(c_{ij}x_j+d_i,c_{ij}\alpha_j+e_{ij}x_j+d_{iL},c_{ij}\beta_j+f_{ij}x_j+d_{iR}),$$

and by (23), the system (9) is equivalent to

$$\left(\sum_{j=1}^n a_{ij}x_j+b_i,\sum_{j=1}^n(a_{ij}\alpha_j+m_{ij}x_j)+b_{iL},\sum_{j=1}^n(a_{ij}\beta_j+n_{ij}x_j)+b_{iR}\right)=$$
$$\left(\sum_{j=1}^n c_{ij}x_j+d_i,\sum_{j=1}^n(c_{ij}\alpha_j+e_{ij}x_j)+d_{iL},\sum_{j=1}^n(c_{ij}\beta_j+f_{ij}x_j)+d_{iR}\right),$$
$$i=1,\ldots,m,$$

or in matrix form,

$$\begin{aligned}&(Ax+b,Ay+Mx+b_L,Az+Nx+b_R)\\&=(Cx+d,Cy+Ex+d_L,Cz+Fx+d_R).\ \Diamond\end{aligned}$$

We have the following result, when all parameters in (27) are nonnegative triangular fuzzy numbers.

Proposition 6: A fuzzy vector $(\tilde{x}_1,\ldots,\tilde{x}_n)^T$ given by $\tilde{x}_j=(x_j,\alpha_j,\beta_j)$, $j=1,\ldots,n,$ is an approximate solution of (9) if and only if it satisfies

$$\begin{cases}(A-C)x-k((M-E)x+(A-C)y)=d-b-k(d_L-b_L),\\(A-C)x+k((N-F)x+(A-C)z)=d-b+k(d_R-b_R),\end{cases}$$

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where $x=(x_1,...,x_n)^T$, $y=(\alpha_1,...,\alpha_n)^T$, $z=(\beta_1,\dots,\beta_n)^T$ and $k=1-\frac{\sqrt{2}}{2}$.

Proof: Since the systems (9) and (30) are equivalent (Proposition 5), the proof is simply obtained by equating the median intervals of left and right sides of (30). ◊

Here, to find nonnegative approximate solutions of the DFLSE which are symmetric triangular fuzzy numbers, we have the following results.

Theorem 2: A fuzzy vector $(\tilde{x}_1,\dots,\tilde{x}_n)^T$ given by $\tilde{x}_j=(x_j,\alpha_j,\alpha_j)$, $j=1,\dots,n$ is a nonnegative approximate symmetric triangular solution of (9) if and only if $x=(x_1,...,x_n)^T$ satisfies the system,

$$\begin{aligned}&(2k(A-C)+(N-F)-(M-E))x\\&=2k(d-b)+d_R-b_R-(d_L-b_L),\end{aligned} \quad (32)$$

and $y=(\alpha_1,...,\alpha_n)^T$ satisfies either

$$(A-C)y=-k(d-b-(A-C)x)+d_L-b_L-(M-E)x, \quad (33)$$

or

$$(A-C)y=k(d-b-(A-C)x)+d_R-b_R-(N-F)x, \quad (34)$$

where $k=2+\sqrt{2}$.

Proof: By solving the system,

$$\begin{cases}(A-C)x-\bar{k}((M-E)x+(A-C)y)=d-b-\bar{k}(d_L-b_L),\\(A-C)x+\bar{k}((N-F)x+(A-C)z)=d-b+\bar{k}(d_R-b_R),\\ \qquad y-z=0,\end{cases} \quad (35)$$

where $\bar{k}=1-\frac{\sqrt{2}}{2}$, we can obtain (32) and (33) or (34). ◊

Proposition 7: Let S_1 , S_{2L} and S_{2R} be the sets of the solutions of (32), (33) and (34), respectively. Then, for all $x\in S_1$, and for all $y\in S_2=S_{2L}\cup S_{2R}$, the fuzzy vector $\tilde{x}=(\tilde{x}_1,\dots,\tilde{x}_n)^T$ with $\tilde{x}_j=(x_j,\alpha_j,\alpha_j)$ is an approximate symmetric solution of the system (9).

Remark 4: Two nonnegative vectors including mean values and spreads are needed to compute a nonnegative approximate symmetric triangular solution of (9). Solving the system (32) results in the mean-value vector, x. If its components are nonnegative, it will be useful to determine the approximate solution. The spread vector, y, is obtained by solving either (33) or (34). One may choose a system which may provide a "better" solution for the problem. For instance, the solution which causes a lower Hausdorff distance between left and right sides of (9) can be considered as a "better" solution. The obtained spreads, y, will be useful if its components are nonnegative and $x_j-y_j\geq 0$, $j=1,...,n$.

The crisp systems of cores and spreads can be solved using any existence method for solving the real linear systems. The obtained results would form a nonnegative approximate symmetric fuzzy solution for (9) if nonnegative solutions of (32), x_j , and nonnegative solutions of the spread system, y_j , satisfy $x_j-y_j\geq 0$, for $j=1,...,n$. Otherwise, we may try to find approximate solutions for (9).

First, we consider the system of cores. Two cases may occur in solving (32).

- The system has some nonnegative solution(s). If the system has more than one nonnegative solution, we choose the solution with minimal Euclidean norm in S_1.
- The system lacks a nonnegative solution. In this case, we consider the following constrained least squares problem,

$$\begin{aligned}&\min_x z_{core}=\|Ux-v\|_2\\&s.t. \qquad\qquad (36)\\&\quad x\geq 0,\end{aligned}$$

where,

$$\begin{aligned}U&=(4+2\sqrt{2})(A-C)+(N-F)-(M-E),\\v&=(4+2\sqrt{2})(d-b)+(d_R-b_R)-(d_L-b_L).\end{aligned} \quad (37)$$

Also, in solving the spread system, two cases may occur.

- The system has some solution(s), y , with all the components being nonnegative and $x_j-y_j\geq 0$, for all $j=1,...,n$. If the system has infinitely many nonnegative solutions, y, which satisfy $x_j-y_j\geq 0$, for all $j=1,...,n$. We choose the solution with minimal Euclidean norm in S_2 .
- The system lacks a desirable solution. In this case, the following constrained least squares problem is considered:

$$\begin{aligned}&\min_y z_{spread}=\|(A-C)y-g\|_2\\&s.t. \qquad\qquad (38)\\&\quad y\geq 0\\&\quad x-y\geq 0,\end{aligned}$$

where, depending on the user's selection between (33) and (34),

$$\begin{aligned}g&=-(2+\sqrt{2})(d-b-(A-C)x)\\&\quad+d_L-b_L-(M-E)x,\end{aligned}$$

or

$$\begin{aligned}g&=(2+\sqrt{2})(d-b-(A-C)x)\\&\quad+d_R-b_R-(N-F)x,\end{aligned}$$

and $y=(\alpha_1,...,\alpha_n)^T$.

Note that problems (36) and (38) always have solutions, since in each case a nonnegative objective functions is minimized on a convex feasible space.

Remark 5: First, we solve the real system (32). If it lacks a nonnegative solution, then we solve the least squares

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problem (36), in which case the value of the objective function in the minimizing problem is non-zero. Then, we solve the real spread system. Similarly, if it lacks the desirable solution, then the least squares problem (38) is solved for which the value of the objective function is non-zero. Ghanbari *et al*. (2010) proposed an approach to guarantee that the obtained solution be fuzzy numbers. Here, we follow the same approach and provide the corresponding necessary adjustments for the optimization problems (36) and (38).

Remark 6: In the least squares problem (36) or (38), we may consider the numbers x_j, for $j=1,\ldots,n$ (or α_j, $j=1,\ldots,n$), with small positive values, say 10^{20}. In this case, the symmetric triangular fuzzy numbers (x_j,α_j,α_j) are very close to zero (or the crisp numbers x_j). Thus, a positive parameter ε can be taken such that $x_j \ge \varepsilon$ (or $\alpha_j \ge \varepsilon$), for all j. This choice guarantees that each fuzzy number has a positive core (or has a positive non-zero spread) at least as much as ε. Therefore, instead of problem (36), we consider the problem,

$$\begin{aligned} &\min_x z_{core} = \|Ux-v\|_2 \\ &s.t. \qquad\qquad (39) \\ &\quad x \ge \varepsilon.u, \end{aligned}$$

and instead of the least squares problem (38), we solve the following optimization problem,

$$\begin{aligned} &\min_y z_{spread} = \|(A-C)y-g\|_2 \\ &s.t. \qquad\qquad (40) \\ &\quad y \ge \varepsilon.u \\ &\quad x-y \ge 0, \end{aligned}$$

where $u=(1,\cdots,1)^T \in \Re^n$ and $\varepsilon > 0$ is a user-defined parameter.

We are ready to give an algorithm for solving the DFLSE regarding the fuzzy multiplication defined in [12].

Algorithm 2: Find an approximate symmetric triangular solution for the DFLSE with the fuzzy multiplication (24).

A. Give $\varepsilon > 0$ to be used for (39) and (40).
B. Solve the system (32). **If** the system (32) has no solution (consequently the system (9) does not have a solution) **or** x, the solution of (32), has at least one negative component **then** solve the optimization problem (39).
C. Find the solution of (33). **If** the system (33) has no solution **or** y, the solution of (33), has at least one negative component **or** there exist an index j such that $x_j - y_j < 0$ **then** solve (34). **If** the system (34) has no solution **or** y, the solution of(34), has at least one negative component **or** there exist an index j such that $x_j - y_j < 0$ **then** solve the optimization problem (40).
D. The approximate symmetric solution of (9) is $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)^T$, where $\tilde{x}_j = (x_j,\alpha_j,\alpha_j)$, $j=1,\ldots,n$, $x=(x_1,\ldots,x_n)^T$ and $y=(\alpha_1,\ldots,\alpha_n)^T$.
E. Stop.

Numerical testing:

Here, to show the effectiveness of our approach in computing an approximate symmetric solution for dual fuzzy linear systems, we apply our proposed methods to several randomly generated test problems. For each test problem, we first produce the coefficient matrices $\tilde{A}=(\tilde{a}_{ij})_{m\times n}$, $\tilde{C}=(\tilde{c}_{ij})_{m\times n}$, and the candidate solution $\tilde{x}=(\tilde{x}_j)_{n\times 1}$, randomly using the function **rand** in Matlab.

Random examples for Algorithm 1

Let $\tilde{a}_{ij}=(a_{ij},b_{ij},c_{ij})$, $\tilde{c}_{ij}=(e_{ij},f_{ij},g_{ij})$ and $\tilde{x}_j=(x_j,y_j,z_j)$, $i=1,\ldots,m$, $j=1,\ldots,n$, be triangular fuzzy numbers and $\tilde{x}_j$ be nonnegative. For all i and j, b_{ij} and f_{ij} are two arbitrary values in $[-k,k]$ and y_j is an arbitrary value in $[0k]$, $k>0$. Moreover, for all i and j, $a_{ij}\in[b_{ij}-s,b_{ij}]$, $c_{ij}\in[b_{ij},b_{ij}+s]$, $e_{ij}\in[f_{ij}-s,f_{ij}]$, $g_{ij}\in[f_{ij},f_{ij}+s]$ and $x_j\in[y_j-s,y_j]$ are arbitrary values where s is a user-defined value and denotes the maximum value of the spreads of the random fuzzy numbers. Also we set $z=2y-x$. Here, we used $k=10$ and $s=2$.

Then, the constant vectors, $\tilde{b}$ and $\tilde{d}$, are specified such that the random vector $\tilde{x}$ satisfies the dual fuzzy system (9) with the fuzzy multiplication (12). The following algorithm presents a method to specify the constant vectors. Note that this algorithm is not the only method to generate the consistent dual fuzzy linear system.

Algorithm 3 : Generate a consistent random generated DFLSE with the multiplication formula (12).

A. Generate the fuzzy matrices $\tilde{A}_{m\times n}=(A,B,C)$, $\tilde{C}_{m\times n}=(E,F,G)$ and $\tilde{x}_{n\times 1}=(x,y,2y-x)$, randomly, according to what mentioned above.
B. Set A^+, A^-, C^+, C^-, E^+, E^-, G^+ and G^-, by Notation 3.2.
C. Set $\alpha=(A^+-E^+)x+(A^--E^-)z$, $\beta=(B-F)y$,

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$\gamma = (C^{-} - G^{-})x + (C^{+} - G^{+})z$, $\lambda = \alpha - \beta$ and $\mu = \gamma - \beta$.

D. Generate a real vector $b_2 \in [-k, k]^m$, randomly, and set $d_2 = b_2 + \beta$.

E. **For** $i = 1, \ldots, m$:

Randomly assign a number in the range of $[0, s]$ to r.

If $\lambda_i \geq 0$ **then** set $b_{1i} = b_{2i} - (\lambda_i + r)$ and $d_{1i} = d_{2i} - r$.

If $\lambda_i < 0$ **then** set $b_{1i} = b_{2i} - r$ and $d_{1i} = d_{2i} - (r - \lambda_i)$.

F. **For** $i = 1, \ldots, m$:

Randomly assign a number in the range of $[0, s]$ to r.

If $\mu_i \geq 0$ **then** set $b_{3i} = b_{2i} + r$ and $d_{3i} = d_{2i} + (\mu_i + r)$. **If** $\mu_i < 0$ **then** set $b_{3i} = b_{2i} + (r - \mu_i)$ and $d_{3i} = d_{2i} + r$.

G. Let $\tilde{b} = (b_1, b_2, b_3)$ and $\tilde{d} = (d_1, d_2, d_3)$.

H. The consistent random generated DFLSE is $(\tilde{A} \otimes_K \tilde{x}) \oplus \tilde{b} = (\tilde{C} \otimes_K \tilde{x}) \oplus \tilde{d}.$

I. **Stop**.

In the following examples, we set $\varepsilon = 10^{-5}$.

Example 1: In this example, the system (18) has appropriate solutions, and thus it would not be necessary to solve the optimization problem (22). The generated test example is:

$$\left\{\begin{array}{l}
((-0.2328, 0.7981, 1.5872) \otimes_K \tilde{x}_1) \oplus ((1.6392, 2.6228, 4.3953) \otimes_K \tilde{x}_2) \oplus \\
((1.2875, 1.4168, 1.9339) \otimes_K \tilde{x}_3) \oplus (-4.8693, 6.6838, 15.0029) = \\
((-1.3140, 0.0768, 1.7493) \otimes_K \tilde{x}_1) \oplus ((-0.3962, 0.6378, 1.5462) \otimes_K \tilde{x}_2) \oplus \\
((-2.5659, -1.4418, 0.0268) \otimes_K \tilde{x}_3) \oplus (49.0819, 49.2381, 49.6741), \\
\\
((-8.7539, -8.0925, -6.8656) \otimes_K \tilde{x}_1) \oplus ((7.0443, 7.1864, 9.0486) \otimes_K \tilde{x}_2) \oplus \\
((9.0646, 9.9370, 11.7327) \otimes_K \tilde{x}_3) \oplus (-26.5314, -9.6871, -8.5439) = \\
((0.8159, 2.2562, 3.7190) \otimes_K \tilde{x}_1) \oplus ((-7.0719, -5.9585, -5.1857) \otimes_K \tilde{x}_2) \oplus \\
((7.9314, 9.3211, 10.1816) \otimes_K \tilde{x}_3) \oplus (-1.4103, -0.0722, 15.5799), \\
\\
((-7.9297, -7.0697, -5.4324) \otimes_K \tilde{x}_1) \oplus ((7.7090, 9.4844, 9.8660) \otimes_K \tilde{x}_2) \oplus \\
((-0.5824, 1.0708, 2.2576) \otimes_K \tilde{x}_3) \oplus (6.2738, 7.2742, 9.9616) = \\
((5.6947, 6.3884, 7.1085) \otimes_K \tilde{x}_1) \oplus ((-1.2351, -0.9221, 0.6290) \otimes_K \tilde{x}_2) \oplus \\
((1.5482, 2.4011, 3.7886) \otimes_K \tilde{x}_3) \oplus (-68.3016, -47.7084, -47.4640).
\end{array}\right.$$

The candidate solution for this system is:

$\tilde{x} = \left((9.0846, 9.4521, 9.8196), \quad (7.8021, 7.8423, 7.8826), \quad (6.8728, 7.0557, 7.2387)\right)^T$.

Solving the system (18), by the function **linsolve** in Matlab, results in:

$w = \left(9.0846, \quad 7.8021, \quad 6.8728, \quad 9.4521, \quad 7.8423, \quad 7.0557\right)^T$,

which satisfies the conditions of the problem (22). Therefore, by setting $x = (w_1, \ldots, w_3)$, $y = (w_4, \ldots, w_6)$ a $z = 2y - x$, the nonnegative approximate symmetric solution is:

$\tilde{x} = \left((9.0846, 9.4521, 9.8196), \quad (7.8021, 7.8423, 7.8826), \quad (6.8728, 7.0557, 7.2387)\right)^T$,

for which the Hausdorff distance from the candidate solution is 2.7534e-14.

Example 2: Consider,

$$\left\{\begin{array}{l}
((-9.9324, -8.5106, -6.6825) \otimes_K \tilde{x}_1) \oplus ((-5.4253, -5.2421, -5.1605) \otimes_K \tilde{x}_2) \oplus \\
((4.2513, 6.0329, 6.1416) \otimes_K \tilde{x}_3) \oplus ((0.0734, 1.2511, 1.3505) \otimes_K \tilde{x}_4) \\
\quad \oplus (-7.5130, -6.2724, 17.9456) = (-57.0449, -54.5866, -53.4957) \oplus \\
((-9.7356, -7.8001, -6.6287) \otimes_K \tilde{x}_1) \oplus ((9.2204, 9.6192, 11.5736) \otimes_K \tilde{x}_2) \oplus \\
((3.5614, 5.2863, 7.1051) \otimes_K \tilde{x}_3) \oplus ((-6.8633, -6.4661, -4.7673) \otimes_K \tilde{x}_4), \\
\\
((-3.5793, -1.6041, -0.2775) \otimes_K \tilde{x}_1) \oplus ((-9.0917, -7.8285, -6.3253) \otimes_K \tilde{x}_2) \oplus \\
((8.7635, 9.2119, 9.3478) \otimes_K \tilde{x}_3) \oplus ((-0.2656, -0.0822, 0.3949) \otimes_K \tilde{x}_4) \\
\quad \oplus (-1.3169, -1.1373, 1.4836) = (-101.7890, -100.9934, -100.9173) \oplus \\
((-1.6759, -1.2914, 0.2413) \otimes_K \tilde{x}_1) \oplus ((0.5501, 1.7642, 3.6363) \otimes_K \tilde{x}_2) \oplus \\
((0.6606, 2.0125, 2.7591) \otimes_K \tilde{x}_3) \oplus ((6.6521, 7.5295, 7.9020) \otimes_K \tilde{x}_4), \\
\\
((-2.7348, -2.0655, -1.0662) \otimes_K \tilde{x}_1) \oplus ((5.2442, 6.0006, 6.6481) \otimes_K \tilde{x}_2) \oplus \\
((3.7211, 5.3150, 6.4620) \otimes_K \tilde{x}_3) \oplus ((-5.4403, -4.9392, -4.6846) \otimes_K \tilde{x}_4) \\
\quad \oplus (-7.6872, -0.2376, 7.1435) = (14.0124, 14.2359, 15.3944) \oplus \\
((7.8418, 8.0376, 8.3868) \otimes_K \tilde{x}_1) \oplus ((-1.3562, 0.0632, 1.8487) \otimes_K \tilde{x}_2) \oplus \\
((2.8817, 4.4517, 4.6843) \otimes_K \tilde{x}_3) \oplus ((-6.0828, -4.9662, -4.5487) \otimes_K \tilde{x}_4), \\
\\
((-2.1817, -1.7414, -0.4718) \otimes_K \tilde{x}_1) \oplus ((-10.8785, -9.1037, -8.9708) \otimes_K \tilde{x}_2) \oplus \\
((0.4481, 2.1646, 3.0430) \otimes_K \tilde{x}_3) \oplus ((-7.2683, -6.0808, -4.8201) \otimes_K \tilde{x}_4) \\
\quad \oplus (6.0715, 7.2226, 10.4643) = (-81.4433, -77.5729, -76.3149) \oplus \\
((-0.0709, 1.3167, 2.3229) \otimes_K \tilde{x}_1) \oplus ((-1.1492, 0.6319, 1.5189) \otimes_K \tilde{x}_2) \oplus \\
((-2.3690, -1.2957, -0.2455) \otimes_K \tilde{x}_3) \oplus ((-3.6406, -2.8703, -1.8095) \otimes_K \tilde{x}_4), \\
\\
((8.7885, 9.4791, 9.6518) \otimes_K \tilde{x}_1) \oplus ((8.3461, 9.1991, 11.0456) \otimes_K \tilde{x}_2) \oplus \\
((8.9266, 9.7808, 9.8356) \otimes_K \tilde{x}_3) \oplus ((-1.1865, -0.1453, 0.8075) \otimes_K \tilde{x}_4) \\
\quad \oplus (3.4058, 4.3291, 5.2930) = (90.2284, 91.5437, 99.4451) \oplus \\
((4.4208, 4.6129, 5.3782) \otimes_K \tilde{x}_1) \oplus ((0.9205, 2.1257, 3.6288) \otimes_K \tilde{x}_2) \oplus \\
((2.9216, 2.9305, 3.6127) \otimes_K \tilde{x}_3) \oplus ((-5.2268, -3.9331, -3.7645) \otimes_K \tilde{x}_4).
\end{array}\right.$$

The candidate solution for this system is:

$$\tilde{x} = \begin{pmatrix} (2.2168, 2.4077, 2.5986) \\ (6.2163, 6.3455, 6.4747) \\ (1.0137, 1.1108, 1.2078) \\ (5.9858, 6.0734, 6.1611) \end{pmatrix}.$$

Solving the system (18), by the function **linsolve** in Matlab, results in:

$w = \left(2.2168, \ 6.2163, \ 1.0137, \ 5.9858, \ 2.4077, \ 6.3455, \ 1.1108, \ 6.0734\right)^T$,

which satisfies the conditions of the problem (22). Thus, the system (18) has appropriate solutions, and it would not be necessary to solve the optimization problem (22). Therefore, by setting $x = (w_1, \ldots, w_4)$, $y = (w_5, \ldots, w_8)$ and $z = 2y - x$, the nonnegative approximate symmetric solution is:

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$$\tilde{x}=\begin{pmatrix}(2.2168,2.4077,2.5986)\\(6.2163,6.3455,6.4747)\\(1.0137,1.1108,1.2078)\\(5.9858,6.0734,6.1611)\end{pmatrix},$$

for which the Hausdorff distance from the candidate solution is 7.3275e-15.

We denote that Algorithm 1 may also be applied to systems with medium scales. Here, we generate the systems in two different ways. Firstly, we consider consistent systems by producing the coefficient matrices and the candidate solutions of these dual fuzzy linear systems, generated randomly and specifying the constant vectors by Algorithm 3. The average Hausdorff distances between the candidate and approximate solutions of some small and medium size consistent systems are shown in Tables 1 and 2 .

Table 1. The results for Algorithm 1 on some small and medium size dual fuzzy linear systems with random candidate solutions, $n=m$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	100
3	NTCSFLinSol	500	250	100
4	AHDCP	2.8550e-13	6.4696e-12	1.3817e-11
5	NTCSVOCLSqP	0	0	0
6	AHDCP	-	-	-
7	AHDOAP	2.8550e-13	6.4696e-12	1.3817e-11

Table 2 . The results for Algorithm 1 on some small and medium size dual fuzzy linear systems with random candidate solutions $n \le m$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	100
3	NTCSFLinSol	500	250	100
4	AHDCP	3.9627e-14	1.7751e-13	8.9293e-7
5	NTCSVOCLSqP	0	0	0
6	AHDCP	-	-	-
7	AHDOAP	3.9627e-14	1.7751e-13	8.9293e-7

Table 3 . The results for Algorithm 1 on some small and medium size dual fuzzy linear systems with random constant vectors, $n=m$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	58	0	0
4	AHDCP	2.6855	-	-
5	NTCSVOCLSqP	442	250	50
6	AHDCP	10.5986	19.9001	21.3637
7	AHDOAP	9.6807	19.9001	21.3637

Secondly, we generate the random dual fuzzy linear systems by producing the coefficient matrices, $\tilde{A}$, $\tilde{C}$ and the constant vectors, $\tilde{b}$, $\tilde{d}$, randomly. Tables 3 and 4 show the average Hausdorff distances between $(\tilde{A}\otimes_K \tilde{x})\oplus\tilde{b}$ and $(\tilde{C}\otimes_K \tilde{x})\oplus\tilde{d}$. In this case, the appearance of inconsistent randomly generated systems is more probable than the generated consistent systems. In the two problem types, the size of the systems are also set randomly; that is, m and n, the number of rows and columns, respectively, are two random numbers.

The first row in the tables shows the range of dimensions of the systems and the second row represents the number of systems randomly generated and solved in each case. Rows 3 and 4 display the number of problems for which the linear system (18) has appropriate solutions, and the corresponding average Hausdorff distances, respectively. The number of problems for which the least squares problem (22) was solved to obtain the symmetric approximate solution for the DFLSE and the average Hausdorff distances are shown in rows 5 and 6, respectively. The average Hausdorff distances over all problems are displayed in the last row. The following abbreviations are used in the tables.

NTCSFLinSol= Number of times the core and spread systems were solved by the function "linsolve".

NTCSVCLSqP= Number of times at least one of the core or spread vectors were obtained by solving the corresponding least squares problems.

AHDCP= Average Hausdorff distance in corresponding category of problems. *AHDOAP*= Average Hausdorff distance over all problems.

Random examples for Algorithm 2

Let $\tilde{a}_{ij}=(a_{ij},m_{ij},n_{ij})$, $\tilde{c}_{ij}=(c_{ij},e_{ij},f_{ij})$ and $\tilde{x}_j=(x_j,y_j,z_j)$, $i=1,...,m$, $j=1,...,n$. For all i, j, we set

$$m_{ij}=\overline{\alpha}a_{ij},\quad n_{ij}=\overline{\alpha}a_{ij},\quad y_j=z_j=\overline{\alpha}x_j,$$

where a_{ij}, c_{ij} and x_j are three arbitrary values in $[0,k]$, $k>0$ and $\overline{\alpha}=\rho.rand$, $\rho\in(0,1]$. Here, we used $\rho=0.25$ and $k=10$. Then, the constant vectors, $\tilde{b}$ and $\tilde{d}$, are specified such that the random vector $\tilde{x}$ satisfies the dual fuzzy system (9) with the fuzzy multiplication (24). The following algorithm presents a method to specify the constant vectors. Note that this algorithm is not the only method to generate the consistent dual fuzzy linear system.

Algorithm 4: Generate a consistent random generated DFLSE with the multiplication formula (24).

A. Generate the nonnegative fuzzy matrices $\tilde{A}_{m\times n}=(A,M,N)$, $\tilde{C}_{m\times n}=(C,E,F)$ and

Table 4. The results for Algorithm 1 on somesmall and medium size dual fuzzy linear systemswith random constant vectors, $n \le m$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	115	0	0
4	AHDCP	9.7349	-	-
5	NTCSVOCLSqP	385	250	50
6	AHDCP	11.2254	20.6219	24.8190
7	AHDOAP	10.8826	20.6219	24.8190

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$\tilde{x}_{n\times 1} = (x, y, y)$, randomly.

B. Set $\alpha = A - C$, $\beta = M - E$, $\gamma = N - F$ and $\delta = \alpha.x$.

C. **For** $i = 1,..., m$:

If $\delta_i \geq 0$ **then** randomly assign a number in the range of $[0,k]$ to b_i and set $d_i = b_i + \delta_i$.

If $\delta_i < 0$ **then** randomly assign a number in the range of $[0,k]$ to d_i and set $b_i = d_i - \delta_i$.

D. Obtain b_L and d_L by solving the following constrained least squares problem,

$$\min \left\| \alpha y + \beta x + b_L - d_L \right\|$$
$$s.t. \qquad (41)$$
$$0 \leq b_L \leq b$$
$$0 \leq d_L \leq d.$$

E. Set $\rho = \alpha y + \gamma x$.

F. **For** $i = 1,..., m$:

If $\rho_i \geq 0$ **then** randomly assign a number in the range of $[0,k]$ to b_{R_i} and set $d_{R_i} = b_{R_i} + \rho_i$.

If $\rho_i < 0$ **then** randomly assign a number in the range of $[0,k]$ to d_{R_i} and set $b_{R_i} = d_{R_i} - \rho_i$.

G. Let $\tilde{b} = (b, b_L, b_R)$ and $\tilde{d} = (d, d_L, d_R)$.

H. The consistent random generated DFLSE is

$$(\tilde{A} \otimes_{DP} \tilde{x}) \oplus \tilde{b} = (\tilde{C} \otimes_{DP} \tilde{x}) \oplus \tilde{d}.$$

I. **Stop**.

Table 5. The results for Algorithm 2 on somesmall and medium size dual fuzzylinear systemswith random candidate solutions, $n = m$ *and* $\rho = 0.2$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	100
3	NTCSFLinSol	497	244	89
4	AHDCP	2.7763e-5	2.3981e-10	1.1080e-11
5	NTCSVOCLSqP	3	6	11
6	AHDCP	0.0387	0.0147	0.0305
7	AHDOAP	2.5980e-4	3.5320e-4	0.0034

In the following example, we set $\varepsilon = 10^{-5}$.

Example 3: In this example, the spread system does not have a nonnegative solution, and so the least squares problem (40) should be solved to find an approximate fuzzy solution of the system,

$$\begin{cases}
((5.9107, 0.3741, 0.0224) \otimes_{DP} \tilde{x}_1) \oplus ((3.7543, 0.4726, 0.0121) \otimes_{DP} \tilde{x}_2) \oplus \\
((6.3328, 0.8537, 0.0955) \otimes_{DP} \tilde{x}_3) \oplus (6.6108, 1.0535, 8.5871) = \\
((6.4680, 1.1849, 0.1230) \otimes_{DP} \tilde{x}_1) \oplus ((3.7364, 0.0696, 0.1255) \otimes_{DP} \tilde{x}_2) \oplus \\
((6.2746, 0.2410, 0.1554) \otimes_{DP} \tilde{x}_3) \oplus (6.6891, 5.6207, 7.6464), \\
\\
((4.4601, 0.2726, 0.1033) \otimes_{DP} \tilde{x}_1) \oplus ((5.4600, 0.7721, 0.1627) \otimes_{DP} \tilde{x}_2) \oplus \\
((9.0541, 0.5719, 0.0623) \otimes_{DP} \tilde{x}_3) \oplus (0.1861, 0.0000, 2.4369) = \\
((3.2410, 0.5604, 0.0667) \otimes_{DP} \tilde{x}_1) \oplus ((7.6674, 0.2077, 0.1531) \otimes_{DP} \tilde{x}_2) \oplus \\
((7.1392, 0.4549, 0.0505) \otimes_{DP} \tilde{x}_3) \oplus (1.3814, 1.3814, 2.6316), \\
\\
((9.2662, 0.6393, 0.1333) \otimes_{DP} \tilde{x}_1) \oplus ((1.1169, 0.1923, 0.1524) \otimes_{DP} \tilde{x}_2) \oplus \\
((6.3056, 1.1045, 0.1742) \otimes_{DP} \tilde{x}_3) \oplus (2.9111, 2.0840, 6.8212) = \\
((6.4035, 0.2562, 0.0138) \otimes_{DP} \tilde{x}_1) \oplus ((1.6809, 0.1855, 0.0808) \otimes_{DP} \tilde{x}_2) \oplus \\
((3.0640, 0.4597, 0.0974) \otimes_{DP} \tilde{x}_3) \oplus (19.4545, 6.1251, 7.9462), \\
\\
((0.9490, 0.0328, 0.1700) \otimes_{DP} \tilde{x}_1) \oplus ((9.0446, 1.7358, 0.1540) \otimes_{DP} \tilde{x}_2) \oplus \\
((0.1424, 0.0249, 0.0135) \otimes_{DP} \tilde{x}_3) \oplus (10.5095, 2.6809, 2.7130) = \\
((8.7975, 0.6770, 0.0123) \otimes_{DP} \tilde{x}_1) \oplus ((5.1973, 0.2221, 0.1088) \otimes_{DP} \tilde{x}_2) \oplus \\
((2.6369, 0.2166, 0.1802) \otimes_{DP} \tilde{x}_3) \oplus (9.7382, 7.2735, 1.3786).
\end{cases}$$

The candidate solution for this system is:

$$\tilde{x} = \left((0.5630, 0.1082, 0.1082), \quad (4.4351, 0.0396, 0.0396), \quad (5.3782, 0.0123, 0.0123)\right)^T.$$

Solving the system (32) by the function **linsolve** in Matlab, results in

$$x = \left(0.5007, 4.3276, 5.4154\right)^T.$$

Solving the systems (33) and (34) results in

$$y = \left(0.3167, 0.4126, -0.1036\right)^T$$

and $y = \left(0.3184, 0.4112, -0.1082\right)^T$,

respectively. Since the vectors had negative components, we then solved the optimization problem (40). Consequently, we obtained the approximate symmetric solution of the dual fuzzy linear system as:

$$\tilde{x} = \left((0.5007, 0.2810, 0.2810), \quad (4.3276, 0.4315, 0.4315), \quad (5.4154, 0, 0)\right)^T,$$

for which the Hausdorff distance from the candidate solution is 0.4995, which is relatively large. When the solutions of (33) and (34) do not satisfy the conditions of the problem (40), we solve the constrained least squares problem (40), so the optimum value of (40) is nonzero, it means that there was a distance between the obtained solution for (40) and the exact solution for the spread system (33) or (34). Therefore, there is a considerable Hausdorff distance between the candidate and approximate solution.

We denote that Algorithm 2 may also be applied to systems with medium scales. Similar to the previous subsection, we generate the systems in two different ways. Firstly, we generate consistent systems by Algorithm 4. The average Hausdorff distances between the candidate and approximate solutions of some small and medium size consistent systems are shown in Tables 5-8. We use two different values for ρ, namely $\rho = 0.2$ and $\rho = 0.8$, to see the effect on the solutions. In the case $\rho = 0.2$ (Tables 5 and 6), almost all systems are

Table 6. The results for Algorithm 2 on some small andmedium size dual fuzzylinear systems with randomcandidate solutions, $n \le m$ and $\rho = 0.2$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	100
3	NTCSFLinSol	487	243	91
4	AHDCP	6.6363e-5	6.2471e-10	9.5218e-13
5	NTCSVOCLSqP	13	7	9
6	AHDCP	0.0368	0.0045	0.0048
7	AHDOAP	0.0010	1.2600e-4	4.3100e-4

Table 7. The results for Algorithm 2 on some small and medium size dual fuzzy linear systems with random candidate solutions, $n = m$ and $\rho = 0.8$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	129	0	2
4	AHDCP	0.2237	-	0.0156
5	NTCSVOCLSqP	371	250	48
6	AHDCP	2.0818	3.8164	3.2296
7	AHDOAP	1.6024	3.8164	3.1010

Table 8. The results for Algorithm 2 on some small and medium size dual fuzzylinear systems with random candidate solutions, $n \le m$ and $\rho = 0.8$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	279	6	0
4	AHDCP	0.2152	0.6112	-
5	NTCSVOCLSqP	221	244	50
6	AHDCP	1.8824	1.2754	2.4470
7	AHDOAP	0.9521	1.2595	2.4470

solved effectively, but when $\rho = 0.8$ (Tables 7 and 8), it is reasonable that some systems turn to be slightly inconsistent and solved by using the corresponding least squares problems, due to increase in the size of the spreads of the fuzzy random numbers.

Secondly, we generate the random dual fuzzy linear systems by producing the coefficient matrices, $\tilde{A}$, $\tilde{C}$ and the constant vectors, $\tilde{b}$, $\tilde{d}$, randomly. Tables 9 and 10 show the average Hausdorff distances between $(\tilde{A} \otimes_{DP} \tilde{x}) \oplus \tilde{b}$ and $(\tilde{C} \otimes_{DP} \tilde{x}) \oplus \tilde{d}$. In this case, the appearance of inconsistent randomly generated systems is more probable than the generated consistent systems. In the two problem types, the size of the systems are also set randomly; that is, m and n, the number of rows and columns, respectively, are two random numbers.

Table 9. The results for Algorithm 2 on some small and medium size dual fuzzylinear systems with random constant vectors, $n = m$ and $\rho = 0.2$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	18	0	0
4	AHDCP	0.3592	-	-
5	NTCSVOCLSqP	482	250	50
6	AHDCP	4.1453	6.9747	7.7179
7	AHDOAP	4.0090	6.9747	7.7179

The first and second rows in the tables are designed similar to ones in the previous subsection. Rows 3 and 4 display the number of problems for which core and spread systems have appropriate solutions, and the corresponding average Hausdorff distances, respectively. The number of problems for which at least one of the core or spread systems were solved by solving the corresponding least squares problems and the average Hausdorff distances are shown in rows 5 and 6, respectively. The average Hausdorff distances over all problems are displayed in the last row. The following abbreviations are used in the tables.

NTCSFLinSol= Number of times the core and spread systems were solved by the function "linsolve".

NTCSVOCLSqP = Number of times at least one of the core or spread vectors were obtained by solving the corresponding least squares problems.

AHDCP= Average Hausdorff distance in corresponding category of problems.

AHDOAP= Average Hausdorff distance over all problems.

Table 10. The results for Algorithm 2 on some small and medium size dual fuzzy linear systems with random constant vectors, $n \le m$ and $\rho = 0.2$.

1	Range of dimensions	[1,10]	[10,100]	[100,200]
2	Number of problems	500	250	50
3	NTCSFLinSol	60	0	0
4	AHDCP	0.0549	-	-
5	NTCSVOCLSqP	440	250	50
6	AHDCP	5.3569	5.9628	6.9313
7	AHDOAP	4.7206	5.9628	6.9313

Conclusions

We considered two classes of dual fuzzy linear systems with respect to different fuzzy multiplication formula, namely, the one defined by Kauffman and Gupta (1991), and the one defined by Dubois and Prade (1980). Median interval defuzzification was applied to obtain an approximate nonnegative symmetric solution for dual fuzzy linear systems. For this purpose, we first transformed an $m \times n$ dual fuzzy linear system into two $m \times n$ crisp systems of equations and solved them. If the crisp systems had no appropriate nonnegative solution, we then proposed to find an approximate solution minimizing the residual Euclidean norm by solving a constrained least squares problem. Moreover, we presented two algorithms for computing approximate nonnegative symmetric solutions of the dual fuzzy linear systems. Finally, two algorithms were presented to generate consistent dual fuzzy linear systems with respect to each multiplication formula followed by extensive numerical results those were provided to show the effectiveness of our approach on randomly generated consistent as well as inconsistent systems.

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