Fixed Point Theorem in Fuzzy Metric Spaces Satisfying E.A Property

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Abstract

The present paper is an attempt to prove common fixed point theorem for six mappings in fuzzy metric spaces. Our main result extends and improves the strict contractive conditions on fuzzy metric spaces and establishes the existence of common fixed points for six mappings under general contractive condition of integral type by using E.A property. The concept of weak compatible mappings is also used to prove the desired result.

Keywords: E.A property, weakly compatible map, contractive condition

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1. Introduction

The notion of a fuzzy set was introduced by (Zadeh, 1965). It brought a turning point in the development of mathematics and laid the foundation of fuzziness in mathematics. Thereafter, it was developed extensively by (Schweizer & Sklar ,1960), (Grabiec , 1988), (Murthy et al., 2010), (Bhatia et al.,2010) and many more authors. This theory also includes interesting applications in diverse areas. (Aamri & Moutawakil, 2002) generalized the concept of non compatibility by defining the notion of E.A property and they proved common fixed point theorems under strict contractive conditions.

An important integral contractive type condition which is used for proving our fixed point theorem is given below:

Branciari-Integral contractive type condition (Branciari, 2000): For a given $\in> 0$, there exists a real number $c \in (0,1)$ and a locally Lebesgue-integrable function $g:[0,\infty) \to [0,\infty)$ such that

$$\int_0^{d(fx,fy)} g(t)dt \le c \int_0^{d(x,y)} g(t)dt \text{ and } \int_0^{\in} g(t)dt > 0 \text{, for all } \forall x, y \in X$$

and for each $\in> 0$.

Remark: Branciari-Integral contractive type condition is generalization of Banach contraction map if $g(t)=1, \forall t \ge 0.$

2. Definitions

Definition 2.1.(Schweizer & Sklar, 1960): A binary operation $*:[0,1] \to [0,1]$ is called continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b \le c*d$, whenever $a \le c$ and $b \le d$; for all $a,b,c,d \in [0,1]$.

Definition 2.2.(Kramosil & Michalek, 1975): The 3-tuple $(X,M,*)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,1]$ satisfying following conditions:

For all $x,y,z \in X$ and $s,t > 0,$

(FM-1) $M(x,y,0)=0$

(FM-2) $M(x,y,t)=1$, for all t>0 if and only if $x=y$

(FM-3) $M(x,y,t)=M(y,x,t)$

(FM-4) $M(x,y,t)*M(y,z,s) \le M(x,z,t+s)$

(FM-5) $M(x,y,.):[0,1) \to [0,1]$ is left continuous.

Here, $M(x,y,t)$ is the degree of nearness between x and y with respect to t.

Definition 2.3.(Grabiec, 1988): Let $(X,M,*)$ be a fuzzy metric space:

(2.3.1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n,x,t)=1$, for all $t>0$.

(2.3.2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$, for all $t>0$ and $p>0$.

(2.3.3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4: A pair of self mapping (f,g) of a fuzzy metric space $(X,M,*)$ is said to be commuting if $M(fgx,gfx,t)=1$; for all $x \in X$.

Definition 2.5: A pair of self mapping (f,g) of a fuzzy metric space $(X,M,*)$ is said to satisfy the E.A property if there exist a sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} M(fx_n,gx_n,t)=1$

Definition 2.6: A pair of self mapping (f,g) of a fuzzy metric space $(X,M,*)$ is said to weakly compatible if they commute at the coincidence points. i.e $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Lemma 2.7.(Mishra et al., 1994)]: Let $(X, M, *)$ be a fuzzy metric space and $x, y \in X, t > 0$ and if for a number $k \in (0,1), M(x,y,kt) \ge M(x,y,t)$ then $x = y$.

3. Main Result

Theorem 3.1: Let (X,M,*) be a fuzzy metric space with continuous t-norm defined by a*b= min {a , b} for all a ,b in [0,1] and A, B, S, T, P and Q be mappings from X into itself such that

$P(X) \subset AB(X), Q(X) \subset ST(X)$

The pair $(P,ST) or (Q,AB)$ satisfies E.A property (3.2)

For all x, y in X, k in (0,1), t>0 such that

$$\int_0^{M(Px,Qy,kt)} \Phi(t)dt \ge \int_0^{m(x,y,t)} \Phi(t)dt \,, \qquad (3.3)$$

Where,

$\Phi : R^+ \to R$ is Lebesgue –integrable mapping which is summable, non negative and such that $\int_0^{\in} \Phi(t)dt > 0$ for each $\in > 0$,

where ,

$$m(x,y,t) = M(STx,Px,t) * M(ABy,Qy,t) * M(ABy,Px,t) * \\ M(STx,Qy,t) * M(STx,ABy,t)$$

for all x, y in X and t>0. If one of P(X), Q(X), ST(X) or AB(X) is complete subspace of X then $(P,ST) and (Q,AB)$ have coincidence point. Further, If $(P,ST) and (Q,AB)$ are weak compatible then P, Q, ST, AB have unique fixed point in X.

Proof: Suppose that the pair (Q, AB) satisfies the E.A. Property. Then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABx_n = l$ for some l in X. (3.4)

Since $Q(X) \subset ST(X)$. So there exists $\{y_n\}$ in X such that

$Qx_n = STy_n$, (3.5)

Hence $\lim_{n\to\infty} STy_n = l$ (3.6)

Now we have to prove that $\lim_{n\to\infty} Py_n = l$

Taking x=y$_n$ and y=x$_n$ in (3.3), then we have

$$\int_0^{M(Py_n,Qx_n,kt)} \Phi(t)dt \ge \int_0^{m(y_n,x_n,t)} \Phi(t)dt \qquad (3.7)$$

Where,

$$m(y_n,x_n,t) = M(STy_n,Py_n,t) * M(ABx_n,Qx_n,t) * M(ABx_n,Py_n,t) \\ * M(STy_n,Qx_n,t) * M(STy_n,ABx_n,t) \,.$$

Taking limit as n→∞ and using (3.4), (3.5), (3.6) and definition of fuzzy metric space, we get $m(y_n,x_n,t) = M(l,Py_n,t)$

By using (3.7), as n→∞, we get

$$\int_0^{M(Py_n,l,kt)} \Phi(t)dt \ge \int_0^{M(l,Py_n,t)} \Phi(t)dt$$

Using lemma (2.7), we get

$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Py_n = l$

Now, suppose that ST(X) is complete subspace of X. Then $l = STr$ for some r in X and subsequently, we get

$$\lim_{n\to\infty} Py_n = \lim_{n\to\infty} STx_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABx_n = l = STr \qquad (3.8)$$

We have to prove that $\Pr = STr$

Taking $x = r, y = x_n$ in (3.3), we have (3.1)

$$\int_0^{M(\Pr,Qx_n,kt)} \Phi(t)dt \ge \int_0^{m(r,x_n,t)} \Phi(t)dt \qquad (3.9)$$

where,

$$m(r,x_n,t) = M(STr,\Pr,t) * M(ABx_n,Qx_n,t) * M(ABx_n,\Pr,t) \\ * M(STr,Qx_n,t) * M(STr,ABx_n,t)$$

By taking n→∞, and using (3.4), (3.5) and (3.6), we get $m(r,x_n,t) = M(STr,\Pr,t)$

Using (3.9), as n→∞, we get $\int_0^{M(\Pr,STr,kt)} \Phi(t)dt \ge \int_0^{M(STr,\Pr,t)} \Phi(t)dt$

By using lemma (2.7), we get $\Pr = STr$

(3.10)

This implies that (P, ST) have coincident point.

The weak compatibility of (P, ST) implies that

$P\Pr = PSTr = ST\Pr = STSTr$ (3.11)

as $P(X) \subset AB(X)$, there exists w in X such that

$\Pr = ABw$ (3.12)

Now we claim that $ABw = Qw$

Now taking x=r and y=w in (3.3), we get

$$\int_0^{M(\Pr,Qw,kt)} \Phi(t)dt \ge \int_0^{m(r,w,t)} \Phi(t)dt \qquad (3.13)$$

Where,

$$m(r,w,t) = M(STr,\Pr,t) * M(ABw,Qw,t) * M(ABw,\Pr,t) \\ * M(STr,Qw,t) * M(STr,ABw,t)$$

Using (3.11) and (3.12), we get $m(r,w,t) = M(\Pr,Qw,t)$

From (3.13), we get $\int_0^{M(\Pr,Qw,kt)} \Phi(t)dt \ge \int_0^{M(\Pr,Qw,t)} \Phi(t)dt$

By using lemma (2.7), this implies that $\Pr = Qw$

Hence, we get $ABw = Qw = \Pr = STr$ (3.14)

The weak compatibility of (Q, AB) implies that

QABw = ABQw = ABABw = QQw (3.15)

Now we shall show that Pr is the common fixed point of $P, Q, STandAB$.

Now taking x=Pr and y=w in (3.3), we get

$$\int_0^{M(P\Pr,Qw,kt)} \Phi(t)dt \ge \int_0^{m(\Pr,w,t)} \Phi(t)dt \qquad (3.16)$$

Where

$$m(\Pr,w,t) = M(STPr,PPr,t) * M(ABw,Qw,t) * M(ABw,P\Pr,t) * M(ST\Pr,Qw,t) * M(ST\Pr,ABw,t)$$

By using (3.11), (3.12) and (3.15), we get

$$m(\Pr,w,t) = M(Qw,P\Pr,t)$$

From (3.16), we get $\int_0^{M(P\Pr,Qw,kt)} \Phi(t)dt \ge \int_0^{M(Qw,P\Pr,t)} \Phi(t)dt$

By using lemma (2.7) and (3.14), we get $P\Pr = Qw = \Pr$ (3.17)

This implies that $P\Pr = \Pr$

Therefore, $\Pr = P\Pr = ST\Pr$ (3.18)

So, Pr is the common fixed point of P and ST.

Similarly, we prove that Qw is the fixed point of Q and AB.

Since, using (3.17), we get $\Pr = Qw$ .So, Pr is the common fixed point of P, Q, AB and ST.

Now, we shall prove the uniqueness of the common fixed point.

If possible, let suppose that u and v are two fixed point of P, Q, AB and ST.

Taking x=u and y=v in (3.3), we get

$$\int_0^{M(Pu,Qv,kt)} \Phi(t)dt \ge \int_0^{m(u,v,t)} \Phi(t)dt \qquad (3.19)$$

Where $m(u,v,t) = M(STu,Pu,t) * M(ABv,Qv,t) * M(ABv,Pu,t) * M(STu,Qv,t) * M(STu,ABv,t)$

This implies that $m(u,v,t) = M(u,v,t)$

Using (3.19), we get $\int_0^{M(u,v,kt)} \Phi(t)dt \ge \int_0^{M(u,v,t)} \Phi(t)dt$

By using lemma (2.7), it follows that u=v

Hence P, Q, AB and ST have a unique fixed point.

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