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Some new types of stabilizers in BL-algebras and their applications

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Abstract

In this paper we introduce some new types of stabilizers in BL-algebras and we state and prove some theorems which determine the relations among stabilizers, MV-algebras and G¨odel algebras. We define the concept ofZRS-condition in BL-algebras and we find a relation between this class of BL-algebras and MV-algebras. Finally, we show that the (semi) normal filters and fantastic filters are equal in BL-algebra.

Keywords: BL-algebra, MV-algebra, G¨odel algebra, fantastic filter, stabilizer filter, normal filter.
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Introduction

BL-algebras are the algebraic structures for H´ajek's Basic logic [7], in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely LukasiewiczLogic, G¨odelLogic and Product Logic. This Basic Logic (BL) is proposed as "the most general" many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. MV-algebra, G¨odel algebra and product algebra are the most known classes of BL- algebras. H´ajekin [7]introduced the notions of filters and prime filters in BL-algebra and by using the prime filters of BL-algebras, he proved the completeness of BL. Filter theory plays an important role in studying these algebras. From logical point of view, various filter correspond to various set of provable formulas. Turunen[12, 13], studied some properties of filters and prime filters of BL-algebra (He called them deductive system and prime deductive system respectively). Now, in this paper we follow [3, 9],and we introduce some new types of stabilizers in BL-algebras and we state and prove some theorems which determine the relationship between stabilizers, MV-algebras and G¨odelalgebras. We prove that $F(L)$ (the set of all filters on BL-algebras L) is a pseudocomplemented lattice. In the follow, we define the concept ofZRS-condition in BL-algebras and we show that, this class of BL-algebras are exactly MV-algebras and by using this condition we prove that the (semi) normal filters are exactly fantastic filters. Finally we answer to open problems that have been appeared in [3].

Preliminaries

Definition 2.1: [7] A BL-algebra is an algebra $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ with four binary operations $\wedge,\vee,\odot,\rightarrow$ and two constant 0,1 such that

(BL1) $(L,\wedge,\vee,0,1)$ is a bounded lattice,
(BL2) $(L,\odot,1)$ is a commutative monoid,
(BL3) $c\leq a\rightarrow b$ b if and only if $a\odot c\leq b$, for all $a,b,c\in L$,
(BL4) $a\wedge b = a\odot(a\rightarrow b)$,
(BL5) $(a\rightarrow b)\vee(b\rightarrow a) = 1$.

A BL- algebra L is called a G¨odelalgebra, if $a^2 = a\odot a = a$, for all $a\in L$ and BL-algebra L is called an MV -algebra, if $(a^*)^* = a$ or equivalently $(a\rightarrow b)\rightarrow b = (b\rightarrow a)\rightarrow a$, for all $a,b\in L$, where $a^* = a\rightarrow 0$.

Lemma 2.2.[4, 5, 7] In each BL-algebra L, the following relations hold for all $a,b,c\in L$:

(BL6) $a\odot b\leq a,b$, $a\odot b\leq a\wedge b$, $a\odot 0 = 0$,
(BL7) $a\leq b$ implies $a\odot c\leq b\odot c$,
(BL8) $a\leq b$ if and only if $a\rightarrow b=1$,
(BL9) $1\rightarrow a=a$, $a\rightarrow a=1$, $a\leq b\rightarrow a$, $a\rightarrow 1=1$,
(BL10) $a\odot a^* = 0$, $1\odot a = a$, $0\rightarrow a=1$,
(BL11) $a\odot b = 0$ if and only if $a\leq b^*$,
(BL12) $a\rightarrow(b\rightarrow c) = (a\odot b)\rightarrow c = b\rightarrow(a\rightarrow c)$,
(BL13) if $a\leq b$ then $b\rightarrow c\leq a\rightarrow c$ and $c\rightarrow a\leq c\rightarrow b$,
(BL14) $a\vee b = ((a\rightarrow b)\rightarrow b))\wedge((b\rightarrow a)\rightarrow a))$,
(BL15) $(a^{**}\rightarrow a)^* = 0$, $(a^{**}\rightarrow a)\vee a^* = 1$.

Definition 2.3.[2, 5, 7, 8, 12] Let F be a non-empty subset of BL- algebra L. Then:

(i) F is called a *filter* of L, if $x,y\in F$ implies $x\odot y\in F$ and $x\in F$, $x\leq y$ imply $y\in F$, for all $x,y\in L$.
(ii) D is called a *deductive system* of L, if $1\in D$ and if $x\in D$ and $x\rightarrow y\in D$, then $y\in D$, for all $x,y\in L$.
(iii) F is called a *fantastic filter*, if $1\in F$ and $z\rightarrow(y\rightarrow x)\in F$ and $z\in F$ imply $((x\rightarrow y)\rightarrow y)\rightarrow x\in F$, for all $x,y,z\in L$.
(iv) A filter F is called a *prime filter*, if $x\vee y\in F$ implies $x\in F$ or $y\in F$, for all $x,y\in L$.
(v) A filter F is called a *normal filter*, if $(y\rightarrow x)\rightarrow x\in F$, then $(x\rightarrow y)\rightarrow y\in F$, for all $x,y\in L$.

Definition 2.4.[6]Let L be a BL-algebra and $X \subseteq L$. The filter of L generated by X will be denoted by $< X >$. We have that $< \emptyset > = 1$ and if $X \neq \emptyset$,

$< X > = \{y \in L \mid x_1 \odot \dots \odot x_n \leq y$ for some $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X\}$.

Theorem 2.5. [8, 11] Let F be a filter of BL-algebra L. Then

(i) every filter of L is a fantastic filter if and only if L is an MV- algebra.

(ii) F is a fantastic filter if and only if $((x \to 0) \to 0) \to x \in F$, for all $x \in L$.

(iii) Let $F \subseteq G$ where F be a fantastic filter and G be a filter. Then G is a fantastic filter.

Theorem 2.6. [7] Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ on L which is defined by

$x \equiv_F y$ if and only if $x \to y \in F$ and $y \to x \in F$

is a congruence relation on L. Define $.\,, \rightharpoonup, \sqcup, \sqcap$ on $\frac{L}{F}$, the set of all congruence classes of L, as follows :

$[x].[y] = [x \odot y]$, $[x] \rightharpoonup [y] = [x \to y]$,

$[x] \sqcup [y] = [x \vee y]$, $[x] \sqcap [y] = [x \wedge y]$

Then $(\frac{L}{F}, ., \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Theorem 2.7.[4,14] Let P be a proper filter of BL-algebra L. Then P is a prime filter if and only if $\frac{L}{P}$ is a BL-chain.

Definition 2.8.[1]An element a of lattice L with bottom element 0 is called an *atom*, if $0 \prec a$ where $x \prec y$ means that $x < y$ and there is no $z \in L \backslash \{0\}$ such that $x < z < y$. An element b of lattice L with top element 1 is called a *co-atom*, if $b \prec 1$.

Definition 2.9. [1]For lattice L whit 0 and $a \in L$, $b \in L$ is called the *pseudocomplemented* of a, if $a \wedge b = 0$ and for each $c \in L$, $c \wedge a = 0$ implies that $c \leq b$. A lattice in which every element has a pseudocomplement is called pseudocomplemented lattice.

Theorem2.10. [11] Let F be a filter of a BL-algebra L. Then we have F is a fantastic filter if and only if the quotient algebra $\frac{L}{F}$ is an MV-algebra.

Note. From now on, in this paper we let $(L, \wedge, \vee, \odot, \to, 0,1)$ or simply L is a BL-algebra, unless otherwise state.

Stabilizer filters in BL- algebras

In [9], the notion of stabilizer X, for any $\emptyset \neq X \subseteq L$, have been defined by $\{a \in L \mid a \to x = x, \forall x \in X\}$ and denoted by $\overline{X}$. But, in this paper we denoted $\overline{X}$ by $\overrightarrow{St}_r(X)$ and we define the other notions such as $\overrightarrow{St}_l(X)$ and $St_\odot(X)$.

Definition 3.1. Let $\emptyset \neq X \subseteq L$. Then *left*, *right* and *product stabilizer* of X is defined as follows:

$\overrightarrow{St}_l(X) = \{a \in L \mid x \to a = x, \forall x \in X\}$

$\overrightarrow{St}_r(X) = \{a \in L \mid a \to x = x, \forall x \in X\}$

$St_\odot(X) = \{a \in L \mid x \odot a = a \odot x = x, \forall x \in X\}$

Note: Let $\emptyset \neq X \subseteq L$, Since by (BL10), $1 \odot x = x \odot 1 = x$, for all $x \in X$, then $1 \in St_\odot(X)$ and so $St_\odot(X) \neq \phi$. Moreover, since by (BL9), $1 \to x = x$, for all $x \in X$, then $1 \in \overrightarrow{St}_r(X)$ and so $\overrightarrow{St}_r(X) \neq \phi$.

Example 3.2.[8](i) Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$ and operations $\odot$ and $\to$ on L are defined as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \to, 0,1)$ is a BL-algebra and

$\overrightarrow{St}_r(\{a\}) = \{1\}$, $\overrightarrow{St}_l(\{a\}) = \emptyset$, $St_\odot(\{b\}) = \{1\}$

(ii) Let $L = \{0, a, b, c, 1\}$, and operations $\wedge$, $\vee$, $\odot$ and $\to$ on L are defined as follows :

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

$\wedge$	0	c	a	b	1
0	0	0	c	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	c	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(L, \wedge, \vee, \odot, \to, 0,1)$ is a BL-algebra and

$\overrightarrow{St}_r(\{a\}) = \{1, b\}$, $\overrightarrow{St}_l(\{1\}) = \{1\}$, $St_\odot(\{b\}) = \{1, b\}$

Proposition3.3. Let $\emptyset \neq X, Y \subseteq L$. Hence,

(i) if $\overrightarrow{St}_l(X) = \{1\}$, then $X = \{1\}$. Also, if $1 \in X$, then $\overrightarrow{St}_l(X) = \{1\}$,

(ii) $\overrightarrow{St}_r(\{1\}) = St_\odot(\{0\}) = L$,

(iii) if $St_\odot(X) = L$, then $X = \{0\}$,

(iv) if $X \subseteq Y$, then $\overrightarrow{St}_r(Y) \subseteq \overrightarrow{St}_r(X)$ and $St_\odot(Y) \subseteq St_\odot(X)$,

(v) $\overrightarrow{St}_r(X) \cup \overrightarrow{St}_r(Y) \subseteq \overrightarrow{St}_r(X \cap Y)$ and $St_\odot(X) \cup St_\odot(Y) \subseteq St_\odot(X \cap Y)$,

(vi) $\overrightarrow{St}_r(X \cup Y) \subseteq \overrightarrow{St}_r(X) \cap \overrightarrow{St}_r(Y)$ and $St_\odot(X \cup Y) \subseteq St_\odot(X) \cap St_\odot(Y)$,

(vii) if F is a filter of L, then $St_\odot(F) = \{1\}$ and $\overrightarrow{St}_l(F) \subseteq F$.

Proof.(i) Let $\overrightarrow{St}_l\ (X)$ = {1}. Then by (BL9), $x = 1 \to x = 1$, for all $x \in X$ and so $X = \{1\}$. Now, let $1 \in X$.Then by (BL9), for any $a \in \overrightarrow{St}_l\ (X)$, $a = 1 \to a = 1$ and so $\overrightarrow{St}_l$ $(X) = \{1\}$.

(ii) By (BL6), $\overrightarrow{St}_r\ (\{1\}) = \{a \in L |\ 1 \to a = 1\} = L$ and by (BL6), $St_\odot(\{0\}) = \{a \in L |\ a \odot 0 = 0\} = L$.

(iii) Let $St_\odot(X) = L$. Since $0 \in L$, then $0 \in St_\odot(X)$ and so for all$x \in X$, $x \odot 0 = 0 \odot x = x$. Now, by (BL6), $x \odot 0 = 0 \odot x = 0$, for all $x \in X$ and so $x = 0$. Hence, $X = \{0\}$.

(iv) Let $X \subseteq Y$and $a \in St_\odot(Y)$. Then $a \odot y = y$, for all $y \in Y$. Since $X \subseteq Y$, then $a \odot y = y$, for all $y \in X$. That is, $a \in St_\odot(X)$. Hence, $St_\odot(Y) \subseteq St_\odot(X)$. By the similar way, we can prove the other case.

(v) Since $X \cap Y \subseteq X, Y$, then by (vi), $\overrightarrow{St}_r\ (X) \subseteq \overrightarrow{St}_r\ (X \cap Y)$ and $\overrightarrow{St}_r\ (Y) \subseteq \overrightarrow{St}_r\ (X \cap Y)$. Therefore, $\overrightarrow{St}_r\ (X) \cup \overrightarrow{St}_r\ (Y)$ $\subseteq \overrightarrow{St}_r\ (X \cap Y)$. By the similar way, $St_\odot(X) \cup St_\odot(Y) \subseteq$ $St_\odot(X \cap Y)$.

(vi) Since $X, Y \subseteq X \cup Y$, then by (vi), $\overrightarrow{St}_r\ (X \cup Y) \subseteq \overrightarrow{St}_r\ (X)$, $\overrightarrow{St}_r\ (Y)$. Hence, $\overrightarrow{St}_r\ (X \cup Y) \subseteq \overrightarrow{St}_r\ (X) \cap \overrightarrow{St}_r\ (Y)$. By the similar way, $St_\odot(X \cup Y) \subseteq St_\odot(X) \cap St_\odot(Y)$.

(vii) Let $a \in St_\odot(F)$. Since $1 \in F$, then $a \odot 1 = 1 \odot a = 1$. Then by (BL10), $a = 1$ and so $St_\odot(F) = \{1\}$.

Now, let $x \in \overrightarrow{St}_l\ (F)$. Then for all $y \in F$, $y \to x = y \in F$. Since F is a filter and$y \in F$, then $x \in F$.

Hence, $\overrightarrow{St}_l\ (F) \subseteq F$.

Theorem3.4. L is a G¨odelalgebra if and only if $\{x\} \subseteq St_\odot(\{x\})$, for all $x \in L$.

proof.$L$is a G¨odelalgebra if and only if $x \odot x = x$, for all $x \in L$, if and only if $x \in St_\odot(\{x\})$, for all $x \in L$.

Theorem 3.5. Let$\emptyset \neq X \subseteq L$, Then $\overrightarrow{St}_r\ (X)$and $St_\odot(X)$are filters of L.

Proof.Let$a, b \in \overrightarrow{St}_r\ (L)$. Then $a \to x = x$and$b \to x = x$, for all $x \in X$. Hence, by (BL7), $(a \odot b) \to x = a \to (b \to x) =$ $a \to x = x$, for all $x \in X$ and so $a \odot b \in \overrightarrow{St}_r\ (X)$. Now, let $a \leq b$ and $a \in \overrightarrow{St}_r\ (X)$.Then $a \to x = x$, for all $x \in X$. Hence , by (BL13),$b \to x \leq a \to x = x$. Since by (BL9), $x \leq b \to x$, then $b \to x = x$and so $b \in \overrightarrow{St}_r\ (X)$. Therefore, $\overrightarrow{St}_r\ (X)$ is a filter of$L$. Now, let $a, b \in St_\odot(X)$. Then $a \odot x = x \odot a = x$and $b \odot x = x \odot b = x$, for all $x \in X$.Since$(L, \odot)$ is a monoid, hence , $(a \odot b) \odot x = a \odot (b \odot x) = a \odot x = x$ and so $a \odot b \in St_\odot(X)$. Finally, let $a \leq b$ and $a \in St_\odot(X)$.

Note: $\overrightarrow{St}_l\ (L)$ is not a filter of L in general.

Example3.6. Let $X = \{b\}$ in the Example 3.2(i). Then $\overrightarrow{St}_l$ $(X) = \{a\}$, which is not a filter of L.

Theorem 3.7. (i) If $a \in L$ is an atom of L, then $St_\odot(\{a\})$is a prime filter.

(ii) If $b \in L$ is a co-atom of$L$, then $\overrightarrow{St}_r\ (\{b\})$is a prime filter.

Proof. (i) Let $a \in L$ be an atom of L. Hence, $a \neq 0$. We claim that $0 \notin St_\odot(\{a\})$. Since if $0 \in St_\odot(\{a\})$, then by (BL6), $0 = 0 \odot a = a \odot 0 = a$, which is impossible. Now, let $x \vee y \in St_\odot(\{a\})$ but $x \notin St_\odot(\{a\})$ and $y \notin St_\odot(\{a\})$, by the contrary. Hence, $(x \vee y) \odot a = a$, but $x \odot a \neq a$ and $y \odot a \neq a$. Now, by (BL6), $x \odot a < a$and $y \odot a < a$. Since a is an atom, then $x \odot a = 0$ and $y \odot a = 0$ and so by (BL11), $x \leq a^*$ and $y \leq a^*$. Hence, $x \vee y \leq a^*$ and so by (BL11), $a = (x \vee y) \odot a = 0$, which is a contradiction. Thus, $x \in St_\odot(\{a\})$ or $y \in St_\odot(\{a\})$ and so $St_\odot(\{a\})$ is a prime filter of L.

(ii) Let $b \in L$ be a co-atom of L. Hence, $b \neq 1$.We claim that $0 \notin \overrightarrow{St}_r\ (\{b\})$. Since if $0 \in \overrightarrow{St}_r\ (\{b\})$, then $0 \to b = b$. Since $0 \leq b$, then $0 \to b = 1$ and so $b = 1$, which is impossible. Now, let $x \vee y \in \overrightarrow{St}_r\ (\{b\})$ but $x \notin \overrightarrow{St}_r\ (\{b\})$ and $y \notin \overrightarrow{St}_r\ (\{b\})$, by the contrary. Then $(x \vee y) \to b = b$, but $x \to b \neq b$ and $y \to b \neq b$. So by (BL9), $b < x \to b$ and $b < y \to b$. Since b is co-atom, then $x \to b = 1$ and $y \to b = 1$ and so $x \leq b$ and $y \leq b$. Hence, $x \vee y \leq b$and so $(x \vee y) \to b = 0$, which is impossible. Therefore,$x \in \overrightarrow{St}_r\ (\{b\})$ or$y \in \overrightarrow{St}_r\ (\{b\})$ and so $\overrightarrow{St}_r\ (\{b\})$is a prim filter of$L$.

Corollary 3.8. If$a \in L$ is an atom and $b \in L$is a co-atom, then$\frac{L}{St\odot(\{a\})}$and$\frac{L}{\overrightarrow{St}_r\ (\{b\})}$are BL-chain.

Proof. By Theorem 2.7 and 3.7, the proof is clear.

Proposition 3.9.[6]Let $F(L)$ be the set of all filters of L. Then $(F(L), \wedge, \vee, \{1\}, L)$ is a bounded complete lattice, where for every family $\{F_i\}_{i \in I}$of filters of L, we have that

$\wedge_{i \in I} F_i = \cap_{i \in I} F_iand\vee_{i \in I} F_i = < \cup_{i \in I} F_i >$

Theorem 3.10. Let F be a filter of L. Then $\overrightarrow{St}_r\ (F)$ is a pseudocomplemented of F.

Proof. First, we prove that $F \cap \overrightarrow{St}_r\ (F) = \{1\}$. Let $x \in F \cap$ $\overrightarrow{St}_r\ (F)$.Since $x \in \overrightarrow{St}_r\ (F)$, then for any $a \in F$, $x \to a = a$. Now, Since$x \in F$, hence for $a = x$ we then $x \to x = x$. But, by (BL9), $x = 1$. Hence $F \cap \overrightarrow{St}_r\ (F) = \{1\}$.Now, let G be a filter of L such that $F \cap G = \{1\}$. Let $a \in G$. Then for any $x \in F$, since $a, x \leq a \vee x$, $x \in F$, $a \in G$, F and $G$are filters of L, then $a \vee x \in F$ and $a \vee x \in G$ and so$a \vee x \in F \cap G = \{1\}$. Hence, $a \vee x = 1$. Now, by (BL14), $((a \to x) \to x) \wedge ((x \to a) \to a) = 1$ and so $(a \to x) \to x = 1$ .Hence, $a \to x \leq x$. Since, by (BL9), $x \leq a \to x$, then $a \to x = x$ and so $a \in \overrightarrow{St}_r\ (F)$.Thus, $G \subseteq$

$\overrightarrow{St}_r\,(F)$.Therefore, $\overrightarrow{St}_r\,(F)$ is a pseudocomplemented of F.

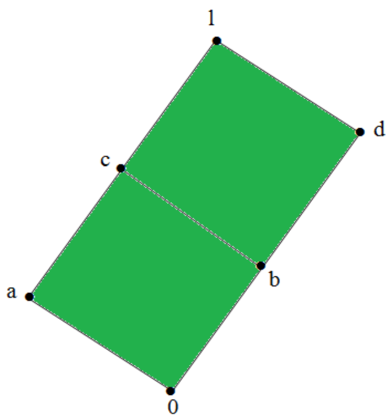
Corollary3.11. $(F(L),\ \wedge,\ \vee,\ \{1\},\ L)$ is a pseudocomplemented lattice.

Proof. By Theorem 3.10, the proof is clear.

BL-algebras with ZRS-condition

Definition4.1 We say BL-algebra L satisfy*Zero Right Stabilizer condition* or briefly*ZRS-condition*if $\overrightarrow{St}_r\,(\{0\}) = \{1\}$.

Example 4.2.[10] Let $L = \{0, a, b, c, d, 1\}$. Then L by the following diagram is a bounded lattice.



Now, let operations "$\rightarrow$" , "*" and "$\odot$" on L are defined as follows :

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

*	0	a	b	c	d	1
	1	d	c	b	a	0

$$x \odot y = min\{z | x \leq y \rightarrow z\} = (x \rightarrow y^*)^*$$

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra.Since $\overrightarrow{St}_r\,(\{0\}) = \{1\}$, then L is a BL-algebra with ZRS-condition.

Example 4.3.BL-algebrain the Example 3.2 (ii), does not satisfy in the ZRS-condition. Since $\overrightarrow{St}_r\,(\{0\}) = \{1, a, b, c\}$.

Theorem 4.4.$L$is an MV-algebraif and only if L satisfies in the ZRS-condition.

Proof. $(\Rightarrow)$ Let L be an MV-algebra and $x \in \overrightarrow{St}_r\,(\{0\})$. Then $x \rightarrow 0 = 0$ and so $x = (x^*)^* = (x \rightarrow 0) \rightarrow 0 = 0 \rightarrow 0 = 1$. Hence, $\overrightarrow{St}_r\,(\{0\}) = \{1\}$.

$(\Leftarrow)$ Let L satisfies in the ZRS-condition. Then $\overrightarrow{St}_r\,(\{0\}) = \{1\}$. Now, let $x \in L$. Then by (BL12)

$x \rightarrow x^{**} = x \rightarrow ((x \rightarrow 0) \rightarrow 0) = (x \rightarrow 0) \rightarrow (x \rightarrow 0) = 1$

Since by (BL15), $(x^{**} \rightarrow x)^* = 0$, then $(x^{**} \rightarrow x) \rightarrow 0 = 0$.

Hence, $x^{**} \rightarrow x \in \overrightarrow{St}_r\,(\{0\}) = \{1\}$.

Therefore,$x^{**} \rightarrow x = 1$ and so $x^{**} = x$. Thus,$L$is an MV-algebra.

Proposition 4.5.Lsatisfiesin the ZRS-condition if and only if for any $x, y \in L$, $x \rightarrow y$ and $y \rightarrow x \in \overrightarrow{St}_r\,(\{0\})$, imply $x = y$.

proof. $(\Rightarrow)$ Let L satisfies in the ZRS-condition, $x, y \in L$ and $x \rightarrow y$ and $y \rightarrow x \in \overrightarrow{St}_r\,(\{0\})$. Since $\overrightarrow{St}_r\,(\{0\}) = \{1\}$, then $x \rightarrow y = y \rightarrow x = 1$, and so $x = y$.

$(\Leftarrow)$ Let $\in \overrightarrow{St}_r\,(\{0\})$. Since by (BL9), $x \rightarrow 1 = 1 \in \overrightarrow{St}_r\,(\{0\})$ and $1 \rightarrow x = x \in \overrightarrow{St}_r\,(\{0\})$, then by hypothesis, $x = 1$. Hence, $\overrightarrow{St}_r\,(\{0\}) = \{1\}$.

Corollary4.6(i) Lsatisfies in the ZRS-condition if and only if every filter of L is a fantastic filter.

(ii) $F$is a fantastic filterif and only if $\frac{L}{F}$satisfies in the ZRS-condition.

(iii) If L satisfies in the ZRS-condition, then for any filter of L,$\frac{L}{F}$satisfies in the ZRS-condition.

Proof.(i) $(\Rightarrow)$Let Lsatisfies in the ZRS-condition. Then by Theorem 4.4, L is an MV-algebra and so by Theorem 2.5 (i), every filter of L is a fantastic filter.

$(\Leftarrow)$ Let every filter of L is a fantastic filter. Then by Theorem 2.5 (i), L is an MV-algebra and so by Theorem 4.4, L satisfies in the ZRS-condition.

(ii)By Theorem 2.10, F be a fantastic filter of L if and only if$\frac{L}{F}$ is an MV-algebra.Also, by Theorem 4.4 ,$\frac{L}{F}$satisfies in the ZRS-condition if and only if $\frac{L}{F}$ is an MV-algebra. Therefore, F is a fantastic filterif and only if$\frac{L}{F}$ satisfies in theZRS-condition.

(iii) Let L satisfies in the ZRS-condition and F be a filter of L. Let $x \in L$ such that $[x] \in \overrightarrow{St}_r\,(\{[0]\})$.Then $[x] \rightharpoonup [0] = [0]$ and so $[x \rightarrow 0] = [0]$ and this means that $(x^*)^* = (x \rightarrow 0) \rightarrow 0 \in F$. Since by Theorem 4.4, L is an MV-algebra , Then $(x^*)^* = x$ and so $x \in F$. Now , since $1 \in F$,then $1 \rightarrow x \in F$ and $x \rightarrow 1 \in F$ and this means that $[x] = [1]$. Hence, $\overrightarrow{St}_r\,(\{[0]\}) = \{[1]\}$.Therefore, $\frac{L}{F}$satisfies in the ZRS-condition.

Note: The converse of Corollary 4.6(iii) is not true in general. Consider following example:
Example 4.7.[8]Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$ and operation $\odot$ and $\rightarrow$ on L are defined as follows :

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then (L, $\wedge$, $\vee$, $\odot$, $\rightarrow$, 0,1) is a BL-algebra and it is clear that $F = \{b, 1\}$is fantastic filter. Now by Theorem 2.10, $\frac{L}{F}$ is MV-algebra (satisfies in the ZRS-condition), but L is not MV-algebra. Since $(b^*)^* = 1$. Therefore, L dose note satisfy in the ZRS-condition.

(Semi) Normal filters and fantastic filters

In this section, we study a new class of filters that called semi normal filter. This is important for us, because this class of filters give a connection between normal filters and fantastic filter. In fact by the above definition, we solved the following open problems in [2]

Open problem 1. [2]Under what suitable conditions, a normal filter becomes a fantastic filter?

Open problem 2.[2] Under what suitable conditions, extension property for normal filter holds?

Definition 5.1. Let F be a nontrivial filter of L. Then F is called a semi-normal filter of , if for all $x \in L$,
$x \in F$if and only if$x^{**} \in F$.

Example 5.2.(i) Let $F = \{1, a, c\}$ in the Example 4.2. Then F is a semi-normal filter. Since $a^{**} = a, c^{**} = c$ and $1^{**} = 1$.
(ii) In any MV-algebra, every filter is a semi-normal filter.

Theorem 5.3. Let F be a normal (fantastic) filter of L. Then F is a semi-normal filter of L.

Proof. Let F be a normal filter and $x \in F$. Since $(x \rightarrow 0) \rightarrow (x \rightarrow 0) = 1$, then by (BL12), $x \rightarrow ((x \rightarrow 0) \rightarrow 0) = 1$ and this means that$x \leq x^{**}$. Since F is a filter and $x \in F$, then $x^{**} \in F$. Now, let$x^{**} \in F$. Hence, $(x \rightarrow 0) \rightarrow 0 \in F$and Since F is normal filter then $(0 \rightarrow x) \rightarrow x \in F$. But , by (BL10), $(0 \rightarrow x) \rightarrow x = x$, hence $x \in F$. Therefore, $F$is a semi-normal filter of L. Now, let F be a fantastic filter of L, and $x \in F$. Similar to the proof of above, $x^{**} \in F$. Now, let $x \in L$ such that $x^{**} \in F$. By Theorem 2.5(ii), $x^{**} \rightarrow x \in F$. Since $x^{**} \in F$and F is a filter of L, then $x \in F$. Hence, F is a semi-normal filter of L.

Theorem 5.4. Let F be a semi-normal filter of L. Then$\frac{L}{F}$ is an MV-algebra.

Proof. Let$F$ be a semi-normal filter of L. Since $F$is a filter of L, then $\frac{L}{F}$is a BL-algebra. We will show that $\frac{L}{F}$ satisfies in the ZRS-condition. Let $[x] \in \overrightarrow{St}_r(\{[0]\})$. Then$[x] \rightarrow [0] = [0]$ and this means that $x^{**} = (x \rightarrow 0) \rightarrow 0 \in F$ and F is a semi-normal filter of L, then $x \in L$ Hence, $x \rightarrow 1 = 1 \in F$ and $1 \rightarrow x = x \in F$ and so $[x] = [1]$. Thus, $\overrightarrow{St}_r(\{[0]\}) = \{[1]\}$ and so $\frac{L}{F}$satisfies in theZRS-condition.Therefore, by Theorem 4.4,$\frac{L}{F}$is a MV-algebra.

Note.Now, in the following Theorem we answer to the open problems that been in [2].

Theorem 5.5. Let F be a filter of L. Then F of L is a normal filter if and only if$F$ is a fantastic filter.

Proof. ($\Longrightarrow$) Let F be normal filter. Then by Theorem 5.3, F is a semi-normal filter and so by Theorem 5.4, $\frac{L}{F}$is an MV-algebra. Now, by Theorem 2.10, $F$is a fantastic filter.
($\Longleftarrow$) Let F be a fantastic filter of L. Then by Theorem 2.10, $\frac{L}{F}$is an MV-algebra. But, in MV-algebras any filter is a normal filter and since $\{[1]\}$ is a filter of$\frac{L}{F}$, then $\{[1]\}$ is a normal filter of$\frac{L}{F}$. Now, we show that F is a normal filter of L . Let $((x \rightarrow y) \rightarrow y) \in F$, for $x, y \in L$. Then $(([x] \rightharpoonup [y]) \rightharpoonup [y]) = [1]$ and so $(([x] \rightharpoonup [y]) \rightharpoonup [y]) \in \{[1]\}$. Since $\{[1]\}$is a normal filter, then $(([y] \rightharpoonup [x]) \rightharpoonup [x]) = [1]$. Hence, $(y \rightarrow x) \rightarrow x \in F$ and this means that F is a normal filter.

Corollary 5.6. (*Extension property)*Let$F$and G be filter of L,$F \subseteq G$and F be a normal filter. Then G is a normal filter

Proof. By Theorems 2.5 (iii) and 5.5, the proof is clear.

Corollary 5.7. Let F be a semi-normal filter of L. Then F is a normal (fantastic) filter.

Proof.Let F be a semi-normal filter of L .Then by Theorems 5.4, $\frac{L}{F}$is a MV-algebra and by Theorem 2.10, F is a fantastic (normal) filter.

Solutions for two open problems in fantastic filters in BL-algebras

In [3], the definitions almost top elements and the set of double complemented elements were studied by*A.BorumandSaeid*in (2009). In that paper there were two open problems for which the answer follows:

Open problem 1. [3] Under which one suitable conditions, if F is a filter of L such that$F = D(F)$, then F is a fantastic filter?

Open problem 2. [3]Under which one suitable conditions, if F is a filter of L such that $(\frac{L}{F}) = \{[1]\}$, then is a fantastic filter ?

For more details, we review some related definitions and theorems.

Definition6.1.[3]Let F be a filter of L. Then
(i) the set of double complemented elements, $D(F)$ is defined by
$D(F) = \{x \in L \mid x^{**} \in F\}$
(ii) An element $x \in L$ is called an almost top element of L, if $x^{**} = 1$.
(iii) We define $N(L)$ as follow:
$N(L) = \{x \in L | x \text{ is an almost top element of } L\}$

$= \{x \in L | x^{**} = 1\} = \{x \in L | x^{*} = 0\}$

Corollary 6.2. $D(\{1\}) = \{x \in L | x^{**} = 1\}$ and $D(\{1\}) = \{x \in L \mid x is an almost top element of L\}$.

Corollary6.3. We can see $N(L)$ is exactly $\overrightarrow{St}_r(\{0\})$ and when $N(L) = \{1\}$, L satisfies in ZRS-condition and so L is a MV-algebra if and only if $N(L) = \{1\}$.

Theorem 6.4.[3]Let F be a filter of$L$. Then

(i) if$F$ is a fantastic filter of L, then $F = D(F)$ and $N(\frac{L}{F}) = \{[1]\}$,

(ii)$F = D(F)$if and only if any almost top element of$\frac{L}{F}$ is trivial.

Theorem 6.5. Let F be a filter of L. Then

(i)If $F = D(F)$, then F is a fantastic filter.

(ii)If $N(\frac{L}{F}) = \{[1]\}$, then F is a fantastic filter.

Proof. (i) Since$F = D(F)$, then by Theorem 6.4 (ii), any almost top element of $\frac{L}{F}$ is trivial. Therefore, $N\left(\frac{L}{F}\right) = \{[1]\}$,and so by Corollary 6.3, $\frac{L}{F}$ is an MV-algebra. Hence, by Theorem 2.10, F is a fantastic filter.

(ii) By (i), the proof is clear.

Conclusion

In this paper we introduced the notion of left, right and product stabilizers in BL-algebras. By this notion we define BL-algebras with ZRS-condition such that they are equal to class MV-algebras and we establish that F is a normal filter if and only if F is a fantastic filter if and only if $F = D(F)$. The results of this paper will be devoted to study the MV-algebra and G¨odel algebra which are different extension of Basic Logic.

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